The topologically twisted index on $S^2 \times T^n$ and black hole entropy

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JTL, L. A. Pando Zayas, V. Rathee and W. Zhao
arXiv:1707.04197, arXiv:1711.01076

Junho Hong and JTL, arXiv:1804.04592

JTL, L. A. Pando Zayas and S. Zhou, in progress

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Supersymmetric partition functions from localization

- Localization is a very powerful tool for computing supersymmetric partition functions and observables
  - S^n partition functions, Wilson loop observables
  - $S^n \times S^1$ partition functions and supersymmetric indices
- Generically, the partition function takes the form

$$Z_{\mathrm{susy}} = \int d\Phi\, Z_{\mathrm{classical}}\, Z_{1-\mathrm{loop}}\, Z_{\mathrm{non-perturbative}}$$

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$$Z_{\rm susy} = \int d\Phi \, Z_{\rm classical} \, Z_{\rm 1-loop} \, Z_{\rm non-perturbative}$$

- We explore the connection between the topologically twisted index on $S^2 \times S^1$ and AdS_4 black hole entropy
  and $S^2 \times T^2$ and AdS_5 black string microstates

[F. Benini, K. Hristov and A. Zaffaroni, arXiv:1511.04085]

Magnetically charged AdS solutions

- AdS/CFT allows us to compare observables on both sides of the duality

Global AdS	$\leftrightarrow$	partition function on S^n
Black holes in AdS	$\leftrightarrow$	partition function on $S^{n-1} \times S^1$
Black strings in AdS	$\leftrightarrow$	partition function on $S^{n-2} \times T^2$

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- Consider a magnetic BPS black hole with spherical horizon

	boundary		near horizon
AdS	AdS_4	$\longrightarrow$	$\mathrm{AdS}_2 \times S^2$
	$\downarrow$		$\downarrow$
CFT	$S^2 \times S^1$	$\longrightarrow$	S^1

The topologically twisted index on $S^2 \times S^1$

- The topologically twisted index was introduced by F. Benini and A. Zaffaroni, arXiv:1504.03698
- Take a magnetic BPS black hole in AdS_4
  What do we do on the field theory side?
  - Background R symmetry flux on S^2
  - This cancels the curvature of $S^2 \Rightarrow$ partial topological twist
  - The index may be computed using localization

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  - The index may be computed using localization
- This topologically twisted index is conjectured to count the black hole microstates [Benini, Hristov, Zaffaroni]
  - Many general features are now known
  - Extended to dyonic black holes, black holes with hyperbolic horizons, magnetic black strings,. . .

Building blocks of the $S^2 \times S^1$ index

- Consider three-dimensional $\mathcal{N} = 2$ Chern-Simons-matter theories on $S^2 \times S^1$
- The index receives contributions from:
  - Vector multiplets:

$$Z_{\text{vector}} = \prod_i \frac{dx_i}{2\pi i x_i} x_i^{k\mathfrak{m}_i} \prod_{\alpha \in G} (1 - x^\alpha)$$

  - Chiral multiplets:

$$Z_{\text{chiral}} = \prod_{\mu \in R} \left(\frac{x^{\mu/2} y^{\mu_f/2}}{1 - x^\mu y^{\mu_f}} \right)^{\mu(\mathfrak{m}) + \mu_f(\mathfrak{n}) - q + 1}$$

- These elements can be combined to construct the index for various models

Counting black hole microstates

- Given a magnetically charged AdS black hole, we can construct the topologically twisted index in the field theory dual and evaluate it in the large-N limit

Counting black hole microstates

- Given a magnetically charged AdS black hole, we can construct the topologically twisted index in the field theory dual and evaluate it in the large-N limit
- We consider the following examples
  1. Magnetic black holes in M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$

     Dual to ABJM theory
  2. Magnetic black holes in massive IIA on $\mathrm{AdS}_4 \times S^6$

     Dual to $\mathcal{N} = 2$ Chern-Simons-matter theory
  3. Magnetic black strings in IIB on $\mathrm{AdS}_5 \times S^5$

     Dual to $\mathcal{N} = 4$ super-Yang-Mills

M-theory on $AdS_4 \times S^7/Z_k$

- The field theory dual is ABJM theory

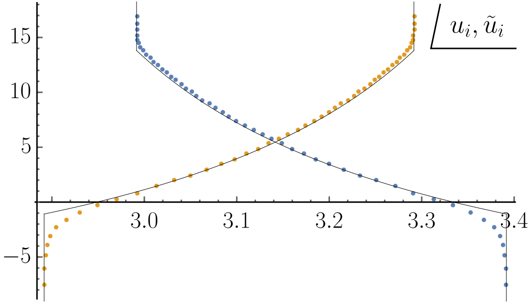
  Chern-Simons-matter with $U(N)_k \times U(N)_{-k}$ gauge groups and bi-fundamental matter A_i, B_j
- The topologically twisted index is given by

$$Z(y_a, \mathfrak{n}_a) = \frac{1}{(N!)^2} \sum_{\mathfrak{m},\tilde{\mathfrak{m}}} \int \prod_i \frac{dx_i}{2\pi i x_i} x_i^{k\mathfrak{m}_i} \prod_{i\neq j} \left(1 - \frac{x_i}{x_j}\right)$$
$$\int \prod_i \frac{d\tilde{x}_i}{2\pi i \tilde{x}_i} \tilde{x}_i^{-k\tilde{\mathfrak{m}}_i} \prod_{i\neq j} \left(1 - \frac{\tilde{x}_i}{\tilde{x}_j}\right)$$
$$\prod_{i,j} \prod_a \left(\frac{\left(\frac{x_i}{\tilde{x}_j} y_a\right)^{1/2}}{1 - \frac{x_i}{\tilde{x}_j} y_a} \right)^{\mathfrak{m}_i - \tilde{\mathfrak{m}}_j - \mathfrak{n}_a + 1} \prod_{i,j} \prod_b \left(\frac{\left(\frac{\tilde{x}_j}{x_i} y_b\right)^{1/2}}{1 - \frac{\tilde{x}_j}{x_i} y_a} \right)^{\tilde{\mathfrak{m}}_j - \mathfrak{m}_i - \mathfrak{n}_b + 1}$$

- The index can be evaluated from the Jeffrey-Kirwan residue

Eigenvalue distribution

- Single solution to the BAE up to permutations



Solution for $\Delta_a = \{0.3, 0.4, 0.5, 2\pi - 1.2\}$ and $N = 50$

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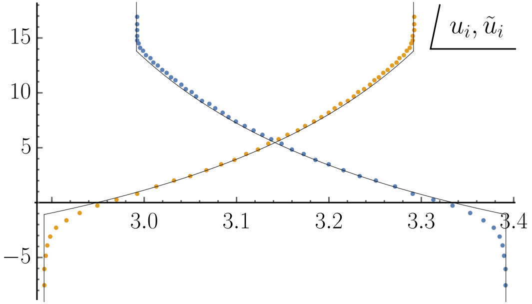

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- Large-N behavior

$$\Re \log Z \sim \boxed{f_0 N^{3/2}} + f_1 N^{1/2} - \frac{1}{2}\log N + \cdots$$

- Subleading terms are difficult to extract analytically

  Tails in the distribution lead to complications

Massive IIA theory on $AdS_4 \times S^6$

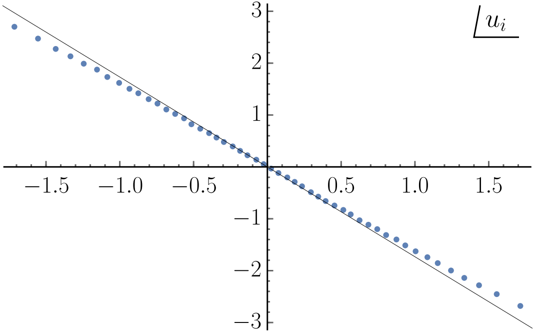
- The dual field theory is an $\mathcal{N} = 2$ Chern-Simons-matter theory with $SU(N)_k$ gauge group and adjoint matter X, Y, Z [Guarino, Jafferis and Varela, arXiv:1504.08009]
- Here the topologically twisted index is given by

$$Z(y_a, \mathfrak{n}_a) = \frac{(-1)^N}{N!} \sum_{\mathfrak{m}} \int \prod_i \frac{dx_i}{2\pi i x_i} x_i^{k\mathfrak{m}_i} \prod_{i \neq j} \left(1 - \frac{x_i}{x_j}\right) \prod_{i,j} \prod_a \left(\frac{\left(\frac{x_i}{x_j} y_a\right)^{1/2}}{1 - \frac{x_i}{x_j} y_a} \right)^{\mathfrak{m}_i - \mathfrak{m}_j + \mathfrak{n}_a + 1}$$

- Once again, the index is evaluated from the Jeffrey-Kirwan residue

Eigenvalue distribution

- Single solution to the BAE up to permutations



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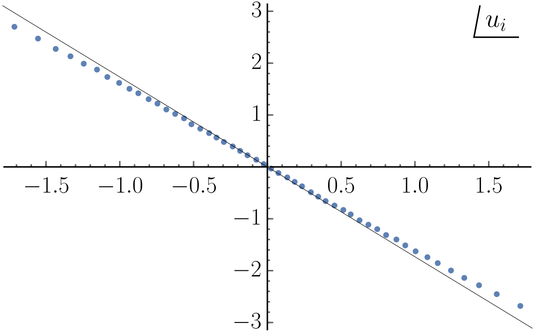

Solution for $\Delta_a = \{0.2, 0.7, 2\pi - 0.9\}$ and $N = 50$

- Large-N behavior

$$\Re \log Z \sim \boxed{f_0 N^{5/3}} + f_1 N^{2/3} + f_2 N^{1/3} + f_3 \log N + \cdots$$

- Can we understand the subleading behavior?

  No tails, but still have to deal with endpoints

Building blocks of the $S^2 \times T^2$ index

- We now turn to the topologically twisted index on $S^2 \times T^2$ where T^2 is parametrized by $q = e^{2\pi i\tau}$
  - Four-dimensional Yang-Mills theory on $S^2 \times T^2$

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- Work in an $\mathcal{N} = 2$ language
  - Vector multiplets:

$$Z_{\text{vector}} = (-1)^{2\rho(\mathfrak{m})} \prod_{i \in G} \frac{dx_i}{2\pi i x_i} \eta(q)^2 \prod_{\alpha \in G} \left(\frac{\theta_1(x^\alpha, q)}{i\eta(q)} \right)$$

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IIB on $AdS_5 \times S^5$

- The dual theory is $\mathcal{N} = 4$ SYM with $SU(N)$ gauge group

  One vector and three chiral multiplets in the $\mathcal{N} = 1$ language
- The topologically twisted index is [Hosseini, Nedelin and Zaffaroni, arXiv:1611.09374]

$$Z(y_a, \mathfrak{n}_a) = \frac{1}{N!} \sum_{\mathfrak{m}} \int \prod_i \frac{dx_i}{2\pi i x_i} \eta(q)^2 \prod_{i \neq j} \left(\frac{\theta_1(\frac{x_i}{x_j}, q)}{i\eta(q)} \right) \prod_{i,j} \prod_a \left(\frac{i\eta(q)}{\theta_1(\frac{x_i}{x_j} y_a, q)} \right)^{\mathfrak{m}_i - \mathfrak{m}_j - \mathfrak{n}_a + 1}$$

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- After evaluating the Jeffrey-Kirwan residue

$$Z = \mathcal{A} \sum_{I \in \mathrm{BAEs}} \frac{1}{\det \mathbb{B}} \prod_{i \neq j} \left[\frac{\theta_1(\frac{x_i}{x_j}, q)}{i\eta(q)} \prod_a \left(\frac{i\eta(q)}{\theta_1(\frac{x_j}{x_j} y_a, q)} \right)^{1 - \mathfrak{n}_a} \right]$$

Solving the BAE

- The BAEs that we need to solve are

$$1 = e^{iB_i} \equiv e^{iv} \prod_j \prod_a \frac{\theta_1(e^{i(u_j - u_i + \Delta_a)}, q)}{\theta_1(e^{i(u_i - u_j + \Delta_a)}, q)}$$

- How do we obtain the u_i's?

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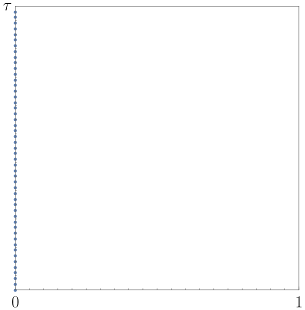
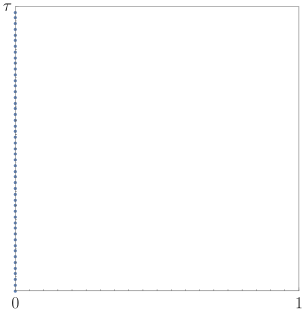
- How do we obtain the u_i's?

  Hosseini, Nedelin, Zaffaroni obtained $u_j = \bar{u} + 2\pi \frac{\tau}{N} j$ in the high-temperature limit $\beta \to 0^+$ where $\tau = i\beta/2\pi$



Multiple solutions to the BAE

- Evenly distributed eigenvalues ⇒ good solution for any q!



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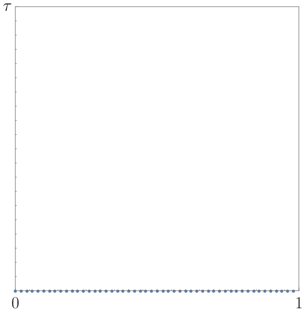

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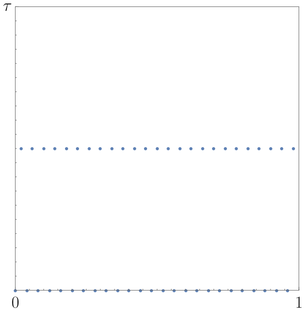

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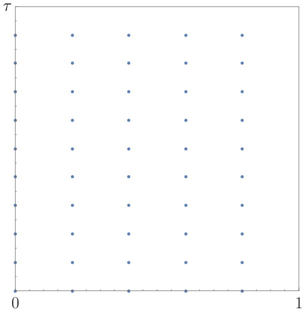

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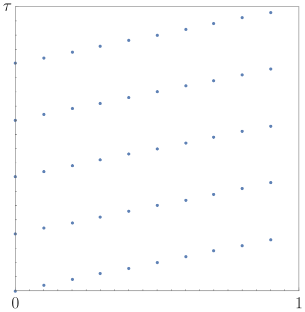

Multiple solutions to the BAE

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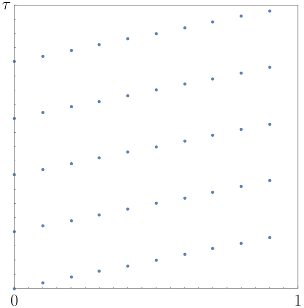

- We find a family of exact solutions specified by $\{m, n, r\}$ where $N = mn$ and $r = 0, 1, \ldots, n-1$

$$u_{jk} = \bar{u} + 2\pi \frac{j + k\tilde{\tau}}{m} \qquad \tilde{\tau} = \frac{m\tau + r}{n}$$

with $j = 0, 1, \ldots, m-1$ and $k = 0, 1, \ldots, n-1$

The topologically twisted index

- The topologically twisted index for $\mathcal{N}=4$ SYM on $S^2\times T^2$ can be written as $Z=\sum_{\{m,n,r\}} Z_{\{m,n,r\}}$ where

$$Z_{\{m,n,r\}}(\Delta_a,\mathfrak{n}_a)=\frac{i^{N-1}}{\det\mathbb{B}_{\{m,n,r\}}}\prod_a\left[\psi(\Delta_a,\tau)\left(\frac{m}{\psi(m\Delta_a,\tilde{\tau})}\right)^N\right]^{1-\mathfrak{n}_a}$$

and

$$\psi(u,\tau)=\frac{\theta_1(u,\tau)}{\eta^3(\tau)}=\sqrt{\varphi_{-2,1}(u,\tau)}$$

Here $\varphi_{-2,1}$ is the unique weak Jacobi form of weight -2 and index 1

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- The sum over sectors is crucial for modularity of the index

 Two modular parameters: $\tau:T^2$ and $\tilde{\tau}:T^2/\mathbb{Z}_m\times\mathbb{Z}_n$

The index as an elliptic genus

- The index computes the elliptic genus of the $\mathcal{N} = (0,2)$ SCFT obtained by reducing on S^2
  - Transforms under $SL(2,\mathbb{Z})$ as a weak Jacobi form of weight 0

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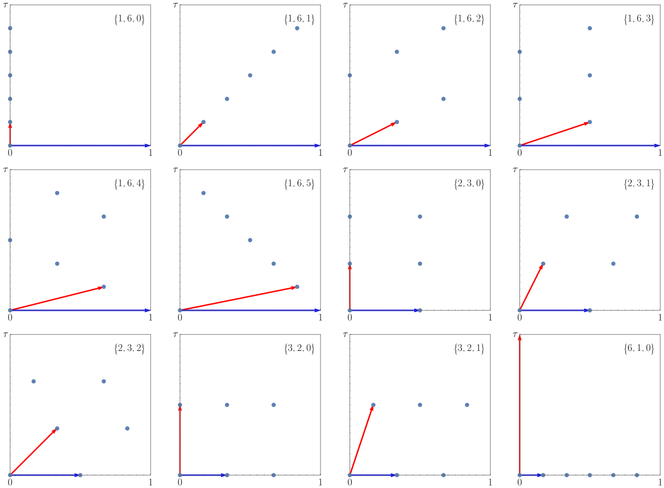
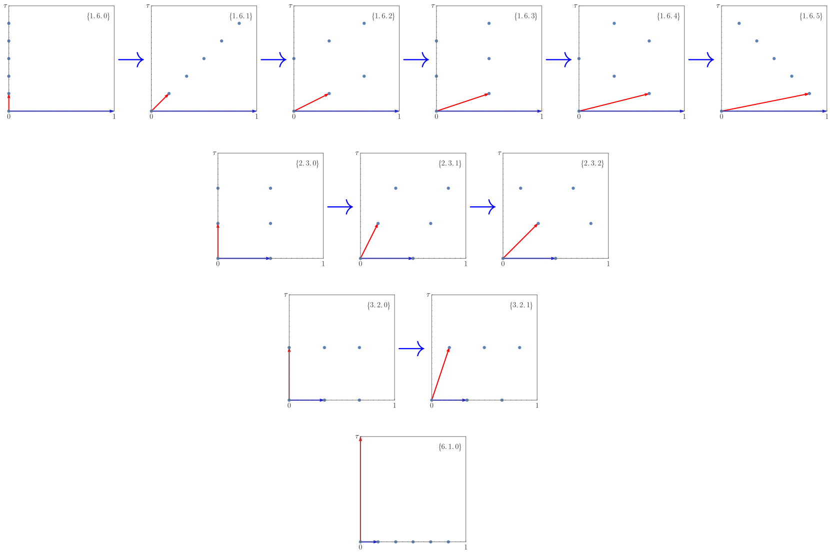
  - Transforms under $SL(2, \mathbb{Z})$ as a weak Jacobi form of weight 0
- Consider, for example, the case $N = 6$

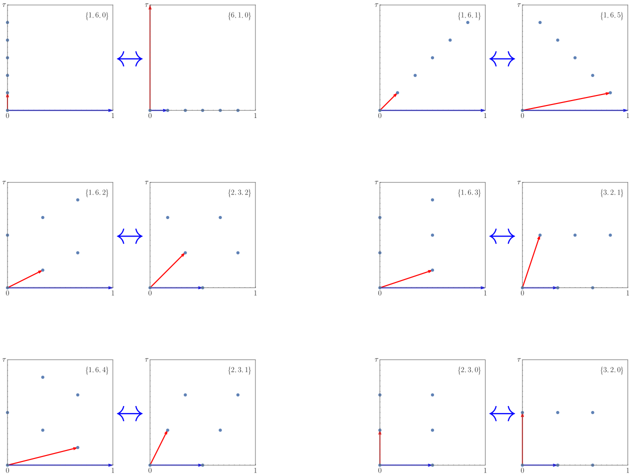
$Z^{N=6} =$



The transformation $T : \tau \to \tau + 1$

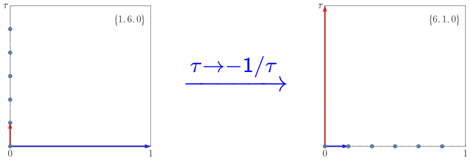


The transformation $S : \tau \to -1/\tau$



The high-temperature limit of the index

- The high-temperature limit $\beta \to 0^+$ where $\tau = i\beta/2\pi$ can be obtained by performing a modular transformation $\tau \to -1/\tau$
- We expect the index to be dominated by a single sector



- This is the sector considered in Hosseini, Nedelin, Zaffaroni

$$\log Z(\Delta_a, \mathfrak{n}_a)\Big|_{\beta\to 0^+} \sim \frac{\pi^2}{6\beta} c_r(\Delta_a, \mathfrak{n}_a)$$

What about the large-N limit?

- In the Cardy limit, we expect N^2 behavior

$$\log Z \sim \frac{N^2}{\beta} \qquad \text{ie} \qquad c_r = \mathcal{O}(N^2)$$

This can be seen in the high-temperature limit of the topologically twisted index

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- Is there a *log*(N) correction?
  And if so, is it universal? Can it be reproduced in the AdS black string dual?

- At finite temperature we expect modular covariance

$$Z \sim N^2 \psi(\Delta_a, \mathfrak{n}_a, \tau)$$

- Can we study the elliptic genus at large-N?
  And on the AdS side of the duality?

Summary

- We have explored the topologically twisted index for theories on $S^2 \times S^1$ and $S^2 \times T^2$
- Main result: There are multiple solutions to the BAE for the index on $S^2 \times T^2$
  - Needed for modular covariance of the index
  - But in the Cardy limit, only a single sector dominates
- Much remains to be understood in the precision counting of AdS black hole microstates

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- Junho Hong will say more in the Gong Show
- Brian McPeak will talk about a separate project on the $D = 6$ index