2-Group Global Symmetry

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References

Based on "Exploring 2-Group Global Symmetry" in collaboration with Dumitrescu and Intriligator

Builds on ideas from "Generalized Global Symmetry" by Gaiotto-Kapustin-Seiberg-Willet

Closely related ideas have been explored in papers by Kapustin-Thorngren, Tachikawa, and Benini-Córdova-Hsin

Basic Problem in Theoretical Physics

Quantum field theories are organized by scale

$$UV \longrightarrow IR$$

Given microscopic constituents and interactions (UV), we wish to solve for the resulting spectrum at long distances (IR)

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Symmetry is one of the few universally applicable tools to constrain RG flows. Suppose that the UV has a global symmetry group G

- The local operators are in G multiplets. This must be reproduced by any effective field theory description at lower energy scales
- If the symmetry is not spontaneously broken, the Hilbert space also forms representations of G. If the symmetry is spontaneously broken the symmetry can be realized non-linearly

Background Gauge Fields and Anomalies

A useful tool for studying global symmetry is to couple to background gauge fields A, leading to a partition function $Z[A]$

The variable A is a fixed classical source. In the case of continuous global symmetry $Z[A]$ is a generating function of correlation functions for the conserved current

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Often, $Z[A]$ is not exactly gauge invariant, but transforms by a local phase

$$Z[A + d\Lambda] = Z[A] \exp(i \int d^d x \, \Lambda f(A))$$

If the phase cannot be removed we say the theory has an 't Hooft anomaly. This is a property of the theory that is constant along the RG trajectory and hence is a powerful constraint on dynamics

Questions

Symmetry and anomalies have a wide range of applications, but there is much still to be learned!

- For continuous global symmetries, their properties are encoded in conserved currents. For discrete symmetries there are no currents. How can we systematically understand and apply anomalies for discrete global symmetries?

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- For continuous global symmetries, their properties are encoded in conserved currents. For discrete symmetries there are no currents. How can we systematically understand and apply anomalies for discrete global symmetries?
- Ordinary global symmetries are characterized by their action on local operators. How can we understand symmetries that act on extended operators like Wilson lines in gauge theories?
- If we incorporate both ordinary global symmetries that act on local operators and generalized and global symmetries that act on extended operators, what possible mixings or non-abelian structures can occur?

Generalized Global Symmetry

A continuous q-form global symmetry is characterized by the existence of a $(q+1)$-form conserved current $J^{(q+1)}$

$$J^{(q+1)}_{A_1 \cdots A_{q+1}} = J^{(q+1)}_{[A_1 \cdots A_{q+1}]} \ , \qquad \partial^{A_1} J^{(q+1)}_{A_1 \cdots A_{q+1}} = 0 \ .$$

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A basic example is $4d$ abelian gauge theory. The Bianchi identity and free equation of motion imply

$$\partial^A \epsilon_{ABCD} F^{CD} = 0 \ , \qquad \partial^A F_{AB} = 0 \ .$$

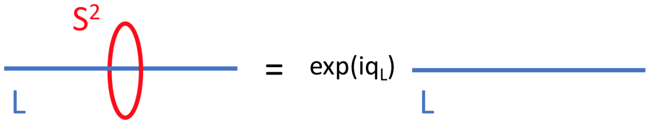
Thus free Maxwell theory has 1-form global symmetry $U(1) \times U(1)$

Charged Line Operators

The charged operators under these symmetries are Wilson and 't Hooft lines. To say that an operator is charged means that if S^2 is a 2-sphere surrounding the line L then

$$\exp\left(i\alpha\int_{S^2} *J^{(2)}\right) L = e^{i\alpha q_L} L$$

In pictures the geometry is



In Maxwell theory this is true since

$$\int_{S^2} *F \sim \text{electric charge}\ , \qquad \int_{S^2} F \sim \text{magnetic charge}$$

Background Fields and Anomalies

Theories with 1-form global symmetry naturally couple to 2-form background gauge fields B

$$\delta S \supset \int d^d x \; B^{CD} J_{CD}$$

Current conservation means (NAIVELY!) that the partition function $Z[B]$ is invariant under background gauge transformations $B^{(2)} \to B^{(2)} + d\Lambda^{(1)}$

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This expectation can be violated by 't Hooft anomalies. For instance in the $4d$ free Maxwell example there is a mixed 't Hooft anomaly between the 1-form global symmetries. This anomaly can be characterized by inflow from a $5d$ Chern-Simons term

$$S_{inflow} = \int d^5 x \; B_e \wedge dB_m$$

Symmetry of QED

Consider $U(1)$ gauge theory with N_f fermions of charge q. What is the symmetry now?

- There is a $SU(N_f)_L \times SU(N_f)_R$ ordinary global symmetry acting on left and right Weyl fermions
- The charged matter means that F is no longer conserved, but $*F$ is still conserved by the Bianchi identity. Thus the 1-form symmetry is $U(1)$.

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Superficially one might expect that ordinary global symmetry and 1-form global symmetry don't talk to each other. In fact they mix. A symptom that something interesting might occur is to examine the related theory where the $U(1)$ is a non-dynamical background field. Then there is an anomalous conservation equation

$$\partial^A J_A \sim qF_L \wedge F_L - qF_R \wedge F_R$$

Making the $U(1)$ field dynamical leads to a new kind of symmetry

2-Group Global Symmetry: Current Algebra

Mixing between the ordinary and 1-form global symmetry encoded in the 3-point function $\langle J_A J_B J_{CD} \rangle$. Analogous to the fact that structure constants for non-abelian ordinary global symmetry are encoded in 3-point functions of J_A

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More precisely, define a 2-group current algebra via Ward identities relating e.g. $\langle J_A J_B J_{CD} \rangle$ and $\langle J_{AB} J_{CD} \rangle$. For simplicity go to a special locus in momentum space $p^2 = q^2 = (p+q)^2 \equiv Q^2$, let M be some scale, C Cartan matrix

$$\langle J_{AB}(p) J_{CD}(-p) \rangle = \frac{1}{p^2} \, \mathtt{f} \left(\frac{p^2}{M^2} \right) (\text{tensor}_{ABCD})$$

$$\langle J_A^i(q) J_B^j(p) J_{CD}(-p-q) \rangle = \frac{\kappa C^{ij}}{2\pi Q^2} \, \mathtt{f} \left(\frac{Q^2}{M^2} \right) (\text{tensor}'_{ABCD})$$

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QED realizes these Ward identities with the constant $\kappa = q$, J^α_A either of the chiral $SU(N_f)$ symmetries, and $J^{(2)} = *F$

2-Group Global Symmetry: Current Algebra

We can think of these Ward identities as arising from a contact term in the OPE of two ordinary currents

$$\partial^A J_A(x) \cdot J_B(0) \sim \frac{\kappa}{2\pi} \partial^C \delta^{(d)}(x) J_{BC}(0)$$

The parameter κ is a quantized structure constant

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Note that contact terms in the OPE of a current are typically associated with charged operators

$$\partial^A J_A(x) \cdot \mathcal{O}(0) \sim i q_{\mathcal{O}} \delta^{(d)}(x) \mathcal{O}(0)$$

In the 2-group OPE above, the derivative on the delta function means that the global charge algebra is unmodified

2-Group Global Symmetry: Background Fields

The Ward identities and OPEs can also be encoded in the properties of background fields

Say both 0-form and 1-form are $U(1)$, so the appropriate background fields are locally

$$\text{1-form gauge field} \quad A^{(1)} , \qquad \text{2-form gauge field} \quad B^{(2)} .$$

Under gauge transformations these mix as

$$A \longrightarrow A^{(1)} + d\lambda^{(0)} , \qquad B^{(2)} \longrightarrow B^{(2)} + d\Lambda^{(1)} + \frac{\kappa}{2\pi}\lambda^{(0)} dA^{(1)} .$$

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This is a Green-Schwarz mechanism for background fields. In mathematics the pair $(A^{(1)}, B^{(2)})$ together with the gluing rule specified via the gauge transformations above form a so-called 2-connection on a 2-group bundle.

An Example with Poincaré Symmetry

One can also have 2-group global symmetry involving Poincaré symmetry and 1-form symmetry. Consider a $U(1)$ gauge theory with four fermions of charges $3, 4, 5, -6$. This theory is consistent. The cubic gauge anomaly vanishes since

$$3^3 + 4^3 + 5^3 = 6^3$$

There is a 1-form global symmetry associated to the current $*F$. The associated 2-form background field $B^{(2)}$ now transforms under local Lorentz transformations

$$B^{(2)} \longrightarrow B^{(2)} + d\Lambda^{(1)} + \frac{\left(\sum q_i\right)}{16\pi} \mathrm{Tr}\left(\theta^{(0)} d\omega^{(1)}\right) ,$$

where $\omega^{(1)}$ is the spin connection and θ is an $SO(4)$ frame rotation parameter

Accidental 2-Group Symmetry

2-group global symmetry can emerge along an RG flow

A simple example is a Georgi-Glashow Model with a fermion Ψ a boson Φ and dynamical $SU(2)$ gauge fields

Field	$SU(N_f)$	$SU(2)$
Ψ	$\square$	$\mathbf{2}$
Φ	$\mathbf{1}$	$\mathbf{3}$

This model is well-defined if N_f is even (Witten anomaly), and UV complete if $N_f \leq 20$

With a suitable potential, this model higgses the $SU(2) \to U(1)$ leading to a 2-group symmetry in the IR

Spontaneously Broken 2-Group Symmetry

2-group global symmetry can be spontaneously broken by the vacuum. Suppose e.g. that both the 0-form and 1-form symmetry are $U(1)$

- The ordinary current J_A leads to a Goldstone boson scalar χ
- The 1-form symmetry current J_{AB} leads to a photon with gauge field strength F

$$J^{(1)} \sim v^2 \, (d\chi) - \frac{i\kappa}{4\pi^2} * (d\chi \wedge F) \ , \qquad J^{(2)} \sim F^{(2)} \ .$$

From this simple expression we can verify the Ward identity relating $\langle J^{(1)} J^{(1)} J^{(2)} \rangle$ and $\langle J^{(2)} J^{(2)} \rangle$.

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Note that the field content of this model is not sensitive to κ. Instead κ enters through an improvement term in $J^{(1)}$. In other words, this model differs from a simple free scalar and free photon through its couplings to background fields

2-Group 't Hooft Anomalies

2-group global symmetry can have 't Hooft anomalies. This means that $Z[A, B]$ is not invariant under 2-group gauge transformations of the pair $(A^{(1)}, B^{(2)})$, but instead transforms by a phase.

Suppose e.g. that the ordinary 0-form symmetry is $U(1)$ and that there is a non-zero triangle diagram with coefficient x involving three currents $J^{(1)}$. (In models of free Weyl fermions $x = \sum_i q_i^3$). This leads to a variation of the partition function under background $A^{(1)}$ gauge transformations

$$Z \to Z \exp\left(\frac{ix}{24\pi^2}\int \lambda dA \wedge dA\right)$$

In models without 2-group symmetry (i.e. $\kappa = 0$) this is the standard chiral anomaly. However when κ is non-zero we must reevaluate whether on not this is an anomaly.

2-Group 't Hooft Anomalies

Consider modifying the partition function by the local counterterm

$$\exp\left(\frac{in}{2\pi}\int B\wedge dA\right)$$

In order to be well-defined under large B gauge transformations, n must be an integer.

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The Green-Schwarz transformation $B \to B + \frac{\kappa}{2\pi}\lambda dA$ means that this counterterm can shift the coefficient x

$$x \to x + 6n\kappa$$

We conclude that only the fractional part of x mod 6κ is an anomaly. The integer part can be absorbed by a local counterterm. This has implications for RG flows. For instance even if there is an anomaly, the IR can be gapped and the anomaly is matched by a non-trivial TQFT

Conclusions

The interplay of ordinary and higher-form global symmetry leads to many rich notions of symmetry, with 2-groups one possibility

There are interesting discrete versions. For instance in $3d$ $U(1)_K$ Chern-Simons theory has $\mathbb{Z}_K$ 1-form global symmetry. If we add matter this often forms a 2-group with other ordinary global symmetry. This machinery is then useful in understanding the dynamics of Chern-Simons matter theories. (Benini-Córdova-Hsin)

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Thanks for Listening!