Thermalization and Random Matrices

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Thermalization of Quantum Systems

How isolated quantum systems thermalize? Systems without additional symmetries – Eigenstate Thermalization Hypothesis

- Individual energy eigenstate is "thermal"

$$\langle E|A|E\rangle \simeq \mathrm{Tr}(\rho_{\mathrm{mic}}A) \simeq \mathrm{Tr}(e^{-\beta H}A)/\mathrm{Tr}(e^{-\beta H})$$

- "Eigenstate Ensemble" explains eventual thermalization

$$\lim_{t\to\infty}\langle\Psi(t)|A|\Psi(t)\rangle = \sum_i |C_i|^2\langle E_i|A|E_i\rangle +$$
$$\lim_{t\to\infty}\sum_{i\neq j} C_i^* C_j\langle E_i|A|E_j\rangle e^{-i(E_i-E_j)t} \simeq A^{\mathrm{th}} + O(1/L)$$

Motivation

- Thermalization after a quantum quench
  AD and Smolkin, arXiv:1709.08654
- ETH in CFT, chaotic CFTs, GGE for 2d CFTs
  AD, Lashkari, Liu, arXiv:1610.00302, arXiv:1611.08764, arXiv:1710.10458
- Collapse of Black Holes as thermalization
- Thermalization in SYK, connection to random matrices and quantum chaos

Eigenstate Thermalization Hypothesis

- ETH ansatz

$$\langle E_i|A|E_j\rangle = A^{\rm eth}(E)\delta_{ij} + \Omega^{-1/2} f(E,\omega) r_{ij}$$

· $E = (E_i + E_j)/2, \;\omega = E_i - E_j$
· $A^{\rm eth}, f$ depend on energy density E/V

Deutsch'91 Srednicki'94; 99 Rigol, Dunjko, Olshanii'08

- Meaning of form-factor $f(\omega)$:

$$\langle A(t)A(0)\rangle_\beta = \int d\omega\, f^2(E,\omega)\, e^{-i\omega t}$$

Chaoticity, ETH and Random Matrices

- Chaotic behavior: Hamiltonian = Random Matrix (WD distribution of energy levels)

- ETH $\simeq$ Eigenstates are random vectors

- "random" behavior of r_{ij}, i.e. A_{ij} with $i \neq j$ (empirical evidence)

- universal "ergodic" behavior of observables $\langle\Psi|A(t)|\Psi\rangle$ for large t (after thermalization)? "structureless" or Haar-invariant A_{ij}
  D'Alessio, Kafri, Polkovnikov, Rigol'15
  Cotler et al., '16, '17

ETH reduces to RMT?

- For small $\omega \le \tau^{-1}$, $f(\omega)$ is constant and r_{nm} is GOE

$$\langle E_i|A|E_j\rangle = A^{\rm eth}\delta_{nm} + \Omega^{-1/2} f(\omega) r_{ij}$$

D'Alessio, Kafri, Polkovnikov, Rigol'15

- Gaussian distribution of r_{ii} and r_{ij}
  Beugeling, Moessner, Haque'14, ...
- ratio $\langle r_{ii}^2\rangle = 2\langle r_{ij}^2\rangle$
  AD and Liu, arxiv:1702.07722, Mondaini, Rigol'17

What is the timescale when ETH reduces to RMT?
Is it $\Delta E_{RMT} = \tau^{-1}$-inverse Thouless time

Thermalization – conventional picture

Diffusive system thermalizes within Thouless time $\tau \sim L^2$ necessary for the slowest diffusive modes to propagate across the system. After time $t \sim \tau$ the system is fully ergodic (and ETH reduces to RMT).

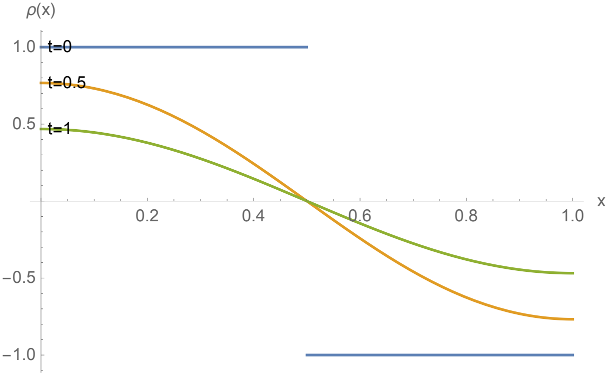
The key idea: dynamics of “slow states” constraints

$$\Delta E_{\mathrm{RMT}}$$

Classical diffusion in 1D

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial x^2}$$

$$\rho(t,x) = \sum_n c_n \cos\left(\frac{\pi n x}{L}\right) e^{-tD(\pi n)^2/L^2}$$



Quasi-classical slow states

- there are states Ψ such that $\langle\Psi|\delta A(t)|\Psi\rangle$ remains of order one long time $t \sim \tau$, where $\delta A = A - A^{\text{eth}}$

$$\langle\Psi|\delta A(t)|\Psi\rangle \sim e^{-t/\tau}$$

- let's consider the deviation $\delta A(t)$ averaged over time T

$$\int dt \langle\Psi|\delta A(t)|\Psi\rangle \frac{\sin(\pi t/T)}{\pi t} \approx \frac{1}{T}\int_0^T dt\langle\Psi|\delta A(t)|\Psi\rangle \sim \frac{\tau}{T}$$

- for any local system $\tau \geq L$, for a diffusive system $\tau \sim L^2$, for non-local SYK system $\tau \sim ???$

From time domain to energy domain and back

Idea: to go from energy domain to time domain

$$\int dt \frac{\sin(\pi t/T)}{t\pi} \langle \Psi(t)|\delta A|\Psi(t)\rangle = \langle \Psi(0)|\delta A_T|\Psi(0)\rangle$$

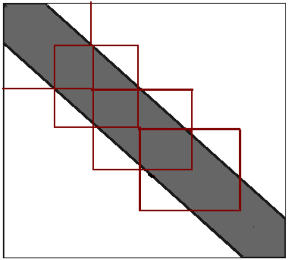
$$(\delta A_T)_{ij} = \begin{cases} \delta A_{ij} : |\omega| \leq 1/T \\ 0 \quad\;\; : |\omega| > 1/T \end{cases} \qquad \delta A_T = \begin{bmatrix} * & \diagdown & & & 0 \\ \diagdown & * & & & \\ & & \cdots & & \swarrow \\ & & & * & \diagdown \\ 0 & & \nearrow\diagdown & & * \end{bmatrix}$$

$$2/T$$

δA_T is a matrix with band structure: within the diagonal band it coincides with A_{ij} with the diagonal $A^{\mathrm{eth}}\delta_{ij}$ part removed, and zero outside

Upper bound on λ of band matrix

- value of $\langle\Psi(0)|\delta A_T|\Psi(0)\rangle$ is bounded by largest eigenvalue $\lambda(\delta A_T)$ of δA_T
- lets introduce $\lambda(\Delta E, E)$ for the largest (by absolute value) eigenvalue of the sub-matrix centered at E and of size $2\Delta E$



$$\lambda(\delta A_T) \leq 2\lambda(E', 2/T) + \lambda(E'', 1/T)$$

Band Random Matrices

- full band matrix δA_T may not be random even when $1/T$ is very small (band is narrow) - because of possible correlations along the diagonal

- by assumption, when $2/T \leq \Delta E_{\rm RMT}$, quadratic sub-matrices of size $\Delta E \leq 2/T$ or smaller are random

- assuming fluctuations r_{ij} are *independent*

$$\lambda^2(\Delta E) \leq 8 \int_0^{1/T} d\omega \, |f(\omega)|^2$$

AD and Liu arxiv:1702.07722

this bound is uniform for all sizes $\Delta E \geq 1/T$ and only depends on the band-width $1/T$

Upper bound on $\Delta E_{\rm RMT}$ from slow states

- for sufficiently large T, such that $T\Delta E_{\rm RMT} \geq 2$

$$\max_{\Psi} \left| \int dt \, \frac{\sin(\pi t/T)}{t \; \pi} \langle \Psi(t) | \delta A | \Psi(t) \rangle \right|^2 \leq \int dt \, \frac{\sin(\pi t/T)}{t \; \pi} \langle A(t) A(0) \rangle_\beta$$

- 2pt function approaches L-independent asymptotic form in the thermodynamic limit $\langle A(t)A(0)\rangle_\beta \sim (t_D/t)^\alpha$

  for 1D diffusive system $\alpha = 1/2$; when the system is finite

$$\int dt \, \frac{\sin(\pi t/T)}{t \; \pi} \langle A(t) A(0) \rangle_\beta \sim \begin{cases} \sqrt{t_D/T} & T \leq \tau \\ \sqrt{t_D \tau}/T & T \geq \tau \end{cases}$$

- taking Ψ to be a slow diffusive mode $\langle \Psi | \delta A(t) | \Psi \rangle \sim e^{-t/\tau}$

$$\left(\frac{\tau}{T}\right)^2 \leq \frac{\sqrt{t_D \tau}}{T} \Rightarrow T \geq L^3$$

Conclusions

- The "Random Matrix" time-scale $\Delta E_{\rm RMT}^{-1}$, when ETH reduces to Random Matrix Theory, is parametrically longer than the Thouless time

- What are the observational signatures of $\Delta E_{\rm RMT}^{-1}$? Is there "ergodicity" and "universality" of $\langle\Psi|A(t)|\Psi\rangle$, or in the end of the story $\Delta E_{\rm RMT}=0$?

  (Hamiltonian = Random Matrix; observable is never random, rather some matrix written in random basis)

- Given that A_{ij} is not "structureless" at Thouless energy scale, what happens at the "end of thermalization" $t\sim\tau$?

What's the Big Picture?

- A new picture of thermalization with the new "Random Matrix" time-scale $\Delta E_{\rm RM}^{-1}$
- The take home point: random matrices are not adequate to describe slow thermalization dynamics. What is the relation between Thouless time defined through spectrum properties and Thouless time defined as thermalization time for many-body systems?
- What are the relevant energy/timescales for the non-local SYK model and what is their bulk interpretation?