A Three dimensional view of the SYK model

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SYK spectrum

GOAL: To outline the steps in order to reproduce the SYK spectrum and bilocal propagator from a three dimensional model and discuss some consequences in the **infinite coupling limit**.

- Consider the SYK hamiltonian for q fermions interacting at a time

$$H = (i)^{q/2} \sum_{1 \leq i_1 < i_2 < \ldots i_q \leq N} J_{i_1 i_2 \ldots i_q} \chi_{i_1} \chi_{i_2} \cdots \chi_{i_q}$$

- On performing the disorder average at large N, define bilocal field, consider $1/N$ fluctuations around saddle point solutions of collective equation of motion, expand the fluctuation field in terms of the modes which diagonalize the quadratic kernel, we obtain the SYK spectrum

$$-(q-1)\frac{\Gamma\left(\frac{3}{2}-\frac{1}{q}\right)\Gamma\left(1-\frac{1}{q}\right)\Gamma\left(\frac{h}{2}+\frac{1}{q}\right)\Gamma\left(\frac{1}{2}+\frac{1}{q}-\frac{h}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{q}\right)\Gamma\left(\frac{1}{q}\right)\Gamma\left(\frac{3}{2}-\frac{1}{q}-\frac{h}{2}\right)\Gamma\left(1-\frac{1}{q}+\frac{h}{2}\right)} = 1$$

[*Maldacena, Stanford*][*Kitaev*]

Bilocal Propagator

- Using the quadratic action for the fluctuation field, the propagator for bilocal fluctuations comes out to be

$$\mathcal{D}(t,z;t',z') \sim \frac{1}{J}\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} \sum_m R(p_m) \frac{Z_{-p_m}(|\omega|z^>) J_{p_m}(|\omega|z^<)}{N_{p_m}}$$

  where

$$R(p_m) = \text{Res}\left(\frac{1}{\tilde{g}(\nu)-1}\right)_{\nu=p_m}$$

  and $\nu = h - 1/2$ $\quad \tilde{g}(\nu) \equiv \frac{1}{g(\nu)}$ and p_m are solutions to the spectral equation.

- For $q=4$, we get

$$R(p_m) = \frac{3p_m^2}{[p_m^2 + (3/2)^2][\pi p_m - \sin(\pi p_m)]}$$

- In the arbitrary q case, $R(p_m)$ can be written in terms of **Harmonic numbers**.

3D model

- For arbitrary q, consider a metric which is conformal to $AdS_2 \times S^1/Z^2$

$$ds^2 = |x|^{4/q-1}\left[\frac{-dt^2+dz^2}{z^2} + \frac{dx^2}{4|x|(1-|x|)}\right]$$

where $-1 < x < 1$.

- For $q = 4$, the conformal factor drops off and we can do a coordinate transformation $x = \sin^2 y$ to get

$$ds^2 = \frac{-dt^2+dz^2}{z^2} + dy^2$$

- Consider a single massive scalar field Φ in this background involving the following non trivial potential

$$V(x) = \frac{1}{|x|^{4/q-1}}\left[4\left(\frac{1}{q}-\frac{1}{4}\right)^2 + m_0^2 + \frac{2V}{J(x)}\left(1-\frac{2}{q}\right)\delta(x)\right]$$

where

$$J(x) = \frac{|x|^{2/q-1}}{2\sqrt{1-|x|}}$$

- Impose Dirichlet boundary conditions on Φ at the ends of the interval, and restrict to fields which are even under $x \to -x$.
- We can write the action for the scalar field as

$$S = \frac{1}{2} \int dt dz dx J(x) \Phi \mathcal{D}_0 \Phi$$

where

$$\mathcal{D}_0 = -\partial_t^2 + \partial_z^2 - \frac{m_0^2}{z^2} + \frac{4}{z^2} \left[x(1-x)\partial_x^2 + \left(\frac{2}{q} - x \left(\frac{1}{2} + \frac{2}{q} \right) \right) \partial_x - \left(\frac{1}{q} - \frac{1}{4} \right)^2 \right]$$

- Expand Φ in terms of a complete basis

$$\Phi(t,z,x) = \int \frac{d\omega dk d\nu}{N_\nu} e^{-i\omega t}|z|^{1/2} Z_\nu(|\omega|z)\phi_k(x)\chi(\omega,\nu,k)$$

where $\phi_k(x)$ are hypergeometic functions which diagonalize the operator in the third direction x. For $q=4$, they are just sine/cosine functions.

- If we choose

$$V = 2(q-1)\frac{\Gamma\left(\frac{3}{2}-\frac{1}{q}\right)\Gamma\left(1-\frac{1}{q}\right)\Gamma\left(\frac{2}{q}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{q}\right)\Gamma\left(2-\frac{2}{q}\right)\Gamma\left(\frac{1}{q}\right)}$$

we can exactly reproduce the SYK spectrum.

Comparison of SYK and 3D propagators

- The 3D propagator evaluated at the center of the interval $\mathcal{G}(t, z, 0; t', z', 0)$ can be written as

$$\sum_m C(p_m, 0, 0) \int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')} \frac{Z_{-p_m}(|\omega| z^>) J_{p_m}(|\omega| z^<)}{N_{p_m}}$$

  where $C(p_m, 0, 0) = \phi_{p_m}(0)\phi_{p_m}(0)$.
- For $q = 4$, we get the following non trivial connection

$$C(p_m, 0, 0) = \frac{2p_m^3}{[p_m^2 + (3/2)^2][\pi p_m - \sin(\pi p_m)]} = \frac{2p_m}{3} R(p_m)$$

- On evaluating the 3D propagator with above $C(p_m, 0, 0)$, we get precise agreement with the SYK propagator.
- For arbitrary q, it can be verified numerically to high accuracy that the 3D propagator agrees with the SYK propagator upto an overall factor which depends **only** on q. Thus, for a given q the 3D propagator and the SYK propagators are proportional to each other.

Large q

- $p_m = 3/2$ remains a solution for all values of q. However, the other solutions simplify at large q [Gross, Rosenhaus], [Maldacena, Stanford]

$$p_m = 2m + \frac{1}{2} + \frac{2}{q}\left(\frac{2m^2+m+1}{2m^2+m-1}\right) + \ldots \qquad m = 1, 2, \ldots$$

- The residue for $p_m = 3/2$ is

$$R(3/2) = \frac{2}{3} - \frac{1}{q}\left(\frac{5}{2} + \frac{\pi^2}{3}\right) + \mathcal{O}\left(\frac{1}{q^2}\right)$$

- The residue for all $p_m \neq 3/2$ are given by

$$R(p_m) \to \frac{1}{q}\frac{4(2m^2+m)}{(2m^2+m-1)^2} + \mathcal{O}\left(\frac{1}{q^2}\right)$$

- Decoupling of these modes while calculating the propagator is consistent with $1/q$ expansion of collective action (Liouville theory)