Dynamics of Near-Extremal Black Holes in AdS_4

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by

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Main results

- The Einstein-Maxwell theory in 4-dimensions doesn't flow to JT theory in IR limit
- However, the dynamics, at low energies and to leading order in the parameter L/r_h, is well approximated by the Jackiw-Teitelboim theory of gravity
- The low-energy dynamics is determined by symmetry considerations alone, with the JT theory being the simplest realisation of these symmetries

Introduction

Problems with $\text{AdS}_2/\text{CFT}_1$

- Degrees of Freedom counting in a d-dimensional theory of gravity:

$$d(d-3)/2$$

tells us that there '-1' degrees of freedom in 2-dimensions [c.f. Finn's talk]!

- A theory with scaling symmetry in time direction has density of states,

$$\rho(E) = A\delta(E) + \frac{B}{E}$$

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- B=0 makes a consistent theory, but it lacks any interesting dynamics!
- How to regulate the backreaction was studied by [Almehiri-Polchinski]

nAdS$_2$/nCFT$_1$

- Recently proposed duality between SYK/tensor model and JT theory [c.f. Sumit's talk, Kitaev, Maldacena-Stanford]
- Polyakov induced gravity theory can also be shown to reproduce the same physics [Mandal-Nayak-Wadia]
- Other models of 2-dimensional gravity can be shown to reproduce the same physics [today's talk :)]
- Symmetry breaking structure:

$$\cancel{\textit{Reparametrization}} \to \mathbb{SL}(2,\mathbb{R})$$

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- Symmetry breaking structure:

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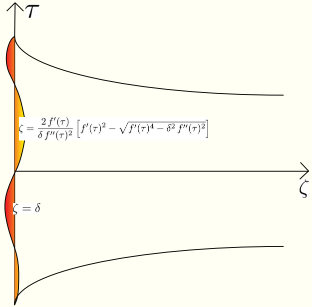
this gives rise to,

$$-\frac{r_h^2}{G}\,\alpha \int_{bdy} \{f(t), t\}$$

Space of AAdS$_2$ Geometries

- 2-dimensional geometries with constant negative curvature and asymptotic AdS boundary conditions can be generated by applying large diffeomorphisms [Mandal-Nayak-Wadia, Jensen]



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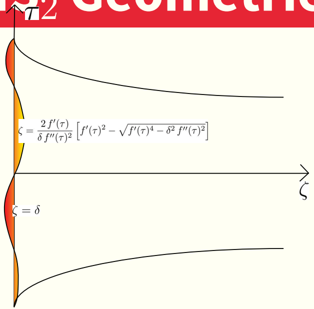

- In Fefferman-Graham gauge, $\delta g_{zz} = 0 = \delta g_{zt}$, these geometries are characterized by metric,

$$ds^2 = \frac{L_2^2}{z^2}\left(dz^2 + dt^2\left(1 - z^2\frac{\{f(t),t\}}{2}\right)^2\right)$$

- These modes as the pseudo-Goldstone modes that Sumit talked about yesterday

Action on AAdS$_2$ geometries

- In models of pure 2-dimensional gravity, these geometries have a trivial action cost associated with them
- when the backreaction is regulated, and reparametrization symmetry is broken the action on these geometries is given by a Schwarzian action,

$$-\frac{r_h^2}{G}\,\alpha\int_{bdy}\{f(t),t\}$$

S-wave Reduction

S-wave Reduction of Einstein-Maxwell

- Einstein-Maxwell system in 4-dimensions

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{\hat{g}}\left(\hat{R} - 2\hat{\Lambda}\right) - \frac{1}{8\pi G}\int d^3x\,\sqrt{\hat{\gamma}}\,K^{(3)} + \frac{1}{4G}\int d^4x\sqrt{\hat{g}}\,F^2$$

can be reduced in the S-wave sector using the following metric ansatz,

$$ds^2 = g_{\alpha\beta}(t,r)\,dx^\alpha dx^\beta + \Phi^2(t,r)\,d\Omega_2^2$$

S-wave Reduction of Einstein-Maxwell

$$S = -\frac{1}{4G}\int d^2x\sqrt{g}\left[2 + \Phi^2(R - 2\hat{\Lambda}) + 2(\nabla\Phi)^2\right] + \frac{2\pi Q_m^2}{G}\int d^2x\sqrt{g}\,\frac{1}{\Phi^2} - \frac{1}{2G}\int_{bdy}\sqrt{\gamma}\,\Phi^2 K.$$

- To compare with the JT action, we need to rescale the 2-dimensional metric,

$$g_{\alpha\beta} \to \tfrac{r_h}{\Phi}\, g_{\alpha\beta}$$

and redefine,

$$\Phi = r_h(1 + \phi)$$

S-wave Reduction of Einstein-Maxwell

- Then the action that one obtains is,

$$
\begin{aligned}
S = & -\frac{r_h^2}{4G}\left(\int d^2x\sqrt{g}\,R + 2\int_{bdy}\sqrt{\gamma}\,K\right) \\
& -\frac{r_h^2}{2G}\int d^2x\,\sqrt{g}\,\phi(R-\Lambda_2) - \frac{r_h^2}{G}\int_{bdy}\sqrt{\gamma}\,\phi K \\
& +\frac{3r_h^2\,\kappa}{G\,L_2^2}\int d^2x\,\sqrt{g}\,\phi^2 - \frac{r_h^2}{2G}\int_{bdy}\sqrt{\gamma}\,\phi^2 K
\end{aligned}
$$

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Does JT still play a role in higher dimensional low energy computation?

4D Spherically Symmetric Reissner-Nordström BH

Solution

- Einstein-Maxwell system has following BH solution:

$$ds^2 = -a(r)^2\,dt^2 + a(r)^{-2}\,dr^2 + b(r)^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2)$$

$$a(r)^2 = 1 - \frac{2GM}{r} + \frac{4\pi Q^2}{r^2} + \frac{r^2}{L^2}$$

$$b(r)^2 = r^2$$

Solution

- Einstein-Maxwell system has following Extremal BH solution:

$$ds^2 = -a(r)^2\,dt^2 + a(r)^{-2}\,dr^2 + b(r)^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2)$$

$$a(r)^2 = \frac{(r - r_h)^2}{r^2 L^2}\left(L^2 + 3r_h^2 + 2rr_h + r^2\right)$$

$$b(r)^2 = r^2$$

$$Q_{ext}^2 = \frac{1}{4\pi}\left(r_h^2 + \frac{3r_h^4}{L^2}\right), \quad M_{ext} = \frac{r_h}{G}\left(1 + \frac{2r_h^2}{L^2}\right)$$

Near Horizon Limit

- The extremal solution has a near horizon AdS_2 limit, $(r - r_h) \ll r_h$:

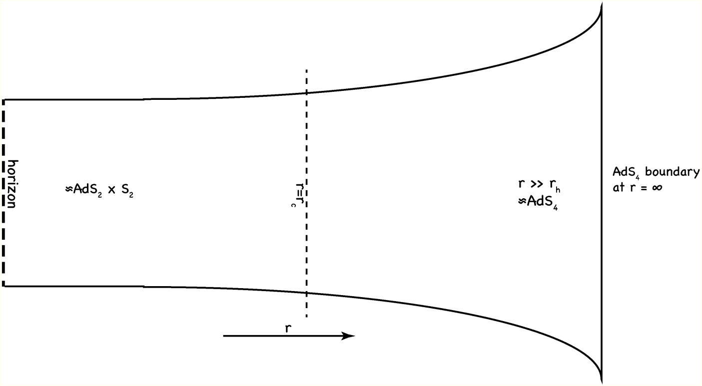
$$ds^2 = \left[-\frac{(r-r_h)^2}{L_2^2}dt^2 + \frac{L_2^2}{(r-r_h)^2}dr^2 + r_h^2\,(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

$$L_2 = \frac{L}{\sqrt{6}}, \text{ is the radius of } AdS_2$$

- Components of the above AdS_2 metric receive corrections @ $\mathcal{O}\left(\frac{r-r_h}{r_h}\right)$
- 'Boundary' of AdS_2 is in the region $(r - r_h) \gg L_2$

$$r_h \gg (r - r_h) \gg L$$

Near Horizon Limit



For $r \to \infty$ the geometry is asymptotically AdS_4.
$r \to r_c$, where $L \ll r_c - r_h \ll r_h$, is the asymptotic $\mathrm{AdS}_2 \times \mathrm{S}^2$ region. The horizon at extremality is at $r = r_h$.

Thermodynamics in Near-Extremal BH

- Extremal Blackholes have 0 temperature
- Heating the BH slightly gives rise to Near-Extremal BH, the degenerate horizon splits into inner and outer horizons,

$$r_\pm = r_h \pm \delta r_h, \quad \delta r_h \ll r_h$$

$$T = \frac{L^2 + 6r_h^2}{2\pi L^2 r_h^2}\delta r_h \to \frac{3}{\pi}\frac{\delta r_h}{L^2}$$

This is achieved by changing the mass of the BH,

$$\delta M = \frac{\delta r_h^2 \left(L^2 + 6r_h^2\right)}{2GL^2 r_h}$$

Thermodynamics in Near-Extremal BH

- Thermodynamic partition function can be computed by evaluating the on-shell action (with correct holographic counterterms [Skenderis-Solodukhin, Balasubramanian-Krauss]),

$$Z[\beta] = e^{-\beta F} = e^{-S-S_{count}}$$

$$S = -\frac{1}{16\pi G}\int_{\mathcal{M}} \sqrt{g}(R-2\Lambda) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} \sqrt{\gamma}\,K + \frac{1}{4G}\int_{\mathcal{M}} \sqrt{g}\,F^2$$

$$S_{count} = \frac{1}{4\pi GL}\int_{\partial\mathcal{M}} \sqrt{\gamma}\left(1+\frac{L^2}{4}R_3\right)$$

Thermodynamics in Near-Extremal BH

- In the generic case, we get,

$$\beta F = \beta M - S_{ent} = \beta M - \frac{\pi r_+^2}{G}$$

- For the near extremal BH, to the leading order

$$\beta F = \beta M_{ext} - \beta \delta M - \frac{\pi r_h^2}{G}$$

- Other thermodynamic quantities:
  - Entropy, $S_{ent} = \frac{\pi r_h^2}{G}$
  - Specific heat, $C = \frac{d\delta M}{dT} = \frac{2\pi^2}{3G} T L^2 r_h$

Comparing with the results of JT

- The action for JT gravity,

$$S_{JT} = -\frac{r_h^2}{4G}\left(\int d^2x\sqrt{g}\,R + 2\int_{bdy}\sqrt{\gamma}\,K\right)$$
$$-\frac{r_h^2}{2G}\left(\int d^2x\sqrt{g}\,\phi\left(R+\frac{2}{L_2^2}\right)+2\int_{bdy}\sqrt{\gamma}\,\phi\left(K-\frac{1}{L_2}\right)\right)$$

- Finite temperature solutions of JT theory are given by,

$$ds^2 = \left(\frac{(r-r_h)^2}{L_2^2}-\frac{2G\delta M}{r_h}\right)d\tau^2 + \frac{dr^2}{\left(\frac{(r-r_h)^2}{L_2^2}-\frac{2G\delta M}{r_h}\right)}$$

  - Topological term $= 4\pi$
  - $R = -2/L_2^2$ therefore the bulk integral doesn't contribute
  - Boundary integral evaluates to $-\beta\delta M$

Comparing with the results of JT

- The action for JT gravity,

$$S_{JT} = -\frac{r_h^2}{4G}\left(\int d^2x\sqrt{g}\,R + 2\int_{bdy}\sqrt{\gamma}\,K\right)$$
$$-\frac{r_h^2}{2G}\left(\int d^2x\sqrt{g}\,\phi\left(R+\frac{2}{L_2^2}\right) + 2\int_{bdy}\sqrt{\gamma}\,\phi\left(K-\frac{1}{L_2}\right)\right)$$

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-

$$S_{JT} = -\beta\delta M - \frac{\pi r_h^2}{G}$$

Computing the 4-pt Function

Minimally-Coupled Bulk Scalar

- $$S = \frac{1}{2}\int d^4x\sqrt{g}\,\left[(\partial\sigma)^2 + m^2\sigma^2\right]$$
- We will be solving the 4-pt function in the weak field approximation
- Using spherical symmetry,

$$\sigma(t,r) = \int d\omega e^{i\omega t}\sigma(\omega,r)$$

the equation of motion for the scalar is,

$$\frac{1}{r^2}\partial_r\left(r^2a^2\partial_r\sigma\right) - \left(\frac{\omega^2}{a^2} + m^2\right)\sigma = 0$$

Minimally-Coupled Bulk Scalar

- In the asymptotic AdS_4 region, the solution for the scalar is,

$$\sigma \sim r^{\Delta_\pm}, \quad \Delta_\pm = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + m^2 L^2}$$

  The source for the dual field theory operator is given by the coefficient of the non-normalizable mode, $\sigma(\omega)$:

$$\sigma \sim \sigma(\omega) \left(\frac{r}{L^2}\right)^{\Delta_+}$$

- AdS/CFT $\Rightarrow$ Classical bulk action = Log[Generating Function for FT]
- Connected part of the FT 4-pt function is given by the term Quartic in $\sigma(\omega)$

Low Frequency Limit of the Correlators

- To consider the contribution from the AdS_2 region, we need to work with the low frequency limit,

$$\omega \sim \frac{r_c - r_h}{L_2^2} \ll \frac{r_h}{L_2^2}$$

- This ensures that outside the AdS_2 throat,

$$\omega \ll \frac{r - r_h}{L_2^2} \Rightarrow \frac{\omega}{m} \ll \frac{r - r_h}{L_2}$$

and, the ω term in the EOM can be dropped!

$$\frac{1}{r^2}\partial_r\left(r^2 a^2 \partial_r \sigma\right) - m^2\sigma = 0$$

- Consequently, the solution for $\sigma(\omega, r) = \sigma(\omega) f(r)$

Correlator Computation

- To consider the contribution of S-wave modes we look at metric perturbations given by,

$$ds^2 = a^2(r)\,(1 + h_{tt})\,dt^2 \;+\; \frac{1}{a^2(r)}(1 + h_{rr})\,dr^2 \;+\; 2h_{tr}\,dt\,dr$$
$$+\; b^2(r)\,(1 + h_{\theta\theta})\,(d\theta^2 \;+\; \sin^2\theta\,d\varphi^2)$$

Gauge fixing: $h_{rr} = 0 = h_{tr}$

- Onshell action is given by,

$$S_{OS} = -\pi \int dt\,dr \left(\frac{b^2}{a^2} h_{tt} T_{tt} + 2h_{\theta\theta} T_{\theta\theta} \right)$$

where, $T_{\mu\nu} = \partial_\mu\sigma\partial_\nu\sigma - \frac{1}{2}\, g_{\mu\nu}\Big[(\partial\sigma)^2 + m^2\sigma^2\Big]$

Correlator Computation

- By integrating out the metric fluctuations, we see,

$$S_{OS} = -8\pi^2 G \int dt \int_{r_h}^{\infty} dr \left(\frac{2a^2b^3}{b'} T_{rr} \frac{1}{\partial_t} T_{tr} - a^2b^2 \left(1 + \frac{2a'b}{b'a}\right) T_{tr} \frac{1}{\partial_t^2} T_{tr} \right)$$

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- In the region where the factorization $\sigma(\omega, r) = \sigma(\omega) f(r)$ holds the contribution of the above expression is just a contact term.

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- In the region where the factorization $\sigma(\omega, r) = \sigma(\omega) f(r)$ holds the contribution of the above expression is just a contact term.
- We can therefore cut-off the radial integral at r_c,

$$S_{OS} = -8\pi^2 G \int dt \int_{r_h}^{r_c} dr \left(\frac{2a^2b^3}{b'} T_{rr} \frac{1}{\partial_t} T_{tr} - a^2b^2 \left(1 + \frac{2a'b}{b'a}\right) T_{tr} \frac{1}{\partial_t^2} T_{tr} \right)$$
$$+ \text{contact terms}$$

Correlator Computation

- A different set of coordinates,

$$z = \frac{L_2^2}{(r - r_h)}$$

$$S_{OS} \simeq 16\pi^2 G \frac{r_h^3}{L_2^2} \int dt \int_{\delta_c}^{\infty} dz\, z \left(T_{tz} \frac{1}{\partial_t^2} T_{tz} - z\, T_{tz} \frac{1}{\partial_t} T_{zz} \right)$$

Comparing with JT

- Recall, that in JT gravity the action reduces to Schwarzian action

$$-\frac{r_h^2}{2G}\left(\int d^2x\sqrt{g}\,\phi\left(R+\frac{2}{L_2^2}\right)+2\int_{bdy}\sqrt{\gamma}\,\phi\left(K-\frac{1}{L_2}\right)\right)$$

$$\Big\downarrow \; ds^2=\frac{L_2^2}{z^2}\left(dz^2+dt^2\left(1-z^2\frac{\{f(t),t\}}{2}\right)^2\right),\quad \phi=\alpha/z$$

$$-\frac{r_h^2}{G}\,\alpha\int_{bdy}\{f(t),t\}$$

Comparing with JT

- For JT coupled to bulk scalar field, for $f(t) = t + \epsilon(t)$

$$S = \frac{r_h L_2^2}{2G} \int dt\, \epsilon(t)\, \epsilon''''(t) \; + \; 4\pi r_h^2 \int dt\, (\epsilon'(t) z T_{zz} + \epsilon(t) T_{tz})$$

which on integrating $\epsilon(t)$ gives,

$$S_{OS} = \frac{16\pi^2 G r_h^3}{L_2^2} \int d^2x \left[z\, T_{tz}\, \frac{1}{\partial_t^2}\, (T_{tz} - z\partial_t T_{zz}) \right]$$

Agrees with the field theory computation!

Conclusions

- Recall that dimensional reduction of the Einstein Maxwell system was different from JT theory,

$$
\begin{aligned}
S =& -\frac{r_h^2}{4G}\left(\int d^2x\sqrt{g}\,R + 2\int_{bdy}\sqrt{\gamma}\,K\right) \\
& -\frac{r_h^2}{2G}\int d^2x\,\sqrt{g}\,\phi(R-\Lambda_2) - \frac{r_h^2}{G}\int_{bdy}\sqrt{\gamma}\,\phi K \\
& +\frac{3r_h^2\,\kappa}{G\,L_2^2}\int d^2x\,\sqrt{g}\,\phi^2 - \frac{r_h^2}{2G}\int_{bdy}\sqrt{\gamma}\,\phi^2 K
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$$+\frac{3r_h^2\,\kappa}{G\,L_2^2}\int d^2x\,\sqrt{g}\,\phi^2 - \frac{r_h^2}{2G}\int_{bdy}\sqrt{\gamma}\,\phi^2 K$$

$$\phi \sim \mathcal{O}\left(\frac{1}{r_h}\right), h_{\mu\nu} \sim \mathcal{O}(1) + \mathcal{O}\left(\frac{1}{r_h}\right)$$

Summary

- The dynamics, at low energies and to leading order in the parameter L/r_h, is well approximated by the Jackiw-Teitelboim theory of gravity
- The low-energy dynamics is determined by symmetry considerations alone, with the JT theory being the simplest realisation of these symmetries
- The fluctuations on the boundary of AdS_2 are related to off-shell gravitons near the horizon of the blackhole
- The dilaton in the 2-dimensional theory, which is related to the size of the compact directions in the higher-dimensional reduction, regulates the backreaction in the AdS_2 region

Future Directions

- Establishing the generality of the results in theories different from Einstein-Maxwell system.
- The microscopic description of extremal BHs is given in terms of the matrix degrees of freedom. We should try to understand how the breaking of time reparametrization symmetry in these models give rise to low energy dynamics.

4-pt Function Computation

Equations of Motion:

$$a^4\partial_r^2 h_{\theta\theta} + a^4\left(\frac{a'}{a}+\frac{3b'}{b}\right)\partial_r h_{\theta\theta} + \frac{a^2}{b^2}\left(1-\frac{8\pi Q^2}{b^2}\right)h_{\theta\theta} = 8\pi G\, T_{tt},$$

$$\left(\frac{a'}{a}-\frac{b'}{b}\right)\partial_t h_{\theta\theta} - \partial_t\partial_r h_{\theta\theta} = 8\pi G\, T_{tr},$$

$$\frac{1}{a^4}\partial_t^2 h_{\theta\theta} + \left(\frac{a'}{a}+\frac{b'}{b}\right)\partial_r h_{\theta\theta} + \frac{b'}{b}\partial_r h_{tt} + \frac{1}{a^2b^2}\left(1-\frac{8\pi Q^2}{b^2}\right)h_{\theta\theta} = 8\pi G\, T_{rr},$$

$$\frac{b^2}{a^2}\partial_t^2 h_{\theta\theta} + a^2b^2(\partial_r^2 h_{\theta\theta}+\partial_r^2 h_{tt}) + 2a^2b^2\left(\frac{a'}{a}+\frac{b'}{b}\right)\partial_r h_{\theta\theta}$$
$$+ a^2b^2\left(\frac{3a'}{a}+\frac{b'}{b}\right)\partial_r h_{tt} + \frac{16\pi Q^2}{b^2}h_{\theta\theta} = 16\pi G\, T_{\theta\theta},$$

4-pt Function Computation

Conservation equations:

$$\frac{1}{a^2}\partial_t T_{tt} + a^2 \partial_r T_{tr} = -2a^2 \left(\frac{a'}{a} + \frac{b'}{b}\right) T_{tr},$$

$$\frac{1}{a^2}\partial_t T_{tr} + a^2 \partial_r T_{rr} = -a^2 \left(\frac{2b'}{b} + \frac{3a'}{a}\right) T_{rr} + \frac{a'}{a^3} T_{tt} + \frac{2b'}{b^3} T_{\theta\theta}.$$

Using these equations we can solve,

$$\partial_r h_{\theta\theta} = \left(\frac{a'}{a} - \frac{b'}{b}\right) h_{\theta\theta} - \partial_t^{-1} \tau_{tr}$$

$$\partial_r h_{tt} = \frac{b}{b'} \left[\tau_{rr} - \frac{1}{a^4}\partial_t^2 h_{\theta\theta} + \frac{a''}{a} h_{\theta\theta} + \left(\frac{a'}{a} + \frac{b'}{b}\right) \partial_t^{-1} \tau_{tr}\right]$$