The OPE of bare twist operators in bosonic S_N orbifold CFTs at large-N

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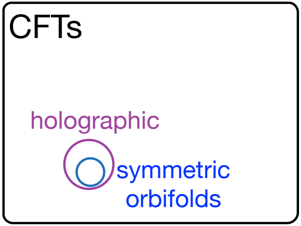
Emergence of AdS bulk from CFT

- Maldacena discovered AdS/CFT holography just over twenty years ago in '97. In '09, Heemskerk-Penedones-Polchinski-Sully showed that it is not actually necessary to have a string/M theory construction involving stacks of $N \gg 1$ D-branes/M-branes in order to obtain a holographic AdS/CFT duality.
- They conjectured that two conditions on a CFT are necessary for producing a holographic AdS bulk dual that is Einstein gravity + small corrections:-
  1. large central charge c, a measure of # d.o.f. of CFT,
  2. sparse spectrum of low-lying operators.

  It is not yet known whether these conditions are sufficient.
- In cases where a string theoretic embedding is available, the above conditions amount to having small g_s and α' corrections.
- More recently, Hartman, Fitzpatrick, Kaplan, various collaborators, and others have studied expansions in $1/c$ to obtain universal results, independent of the specific field content and interactions of the CFT. They also studied h/c expansions, where h is the conformal weight of a heavy operator.
- In the AdS_3/CFT_2 context, keeping h/c fixed as $c \to \infty$ corresponds to keeping the black hole horizon radius fixed while sending $G_N \to 0$. Even nonperturbative $\mathcal{O}(e^{-c})$ physics has been investigated.

Prototypical holographic CFTs

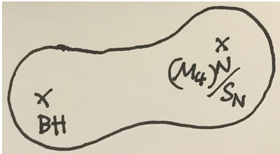
- CFTs can live in 0+1 up to 5+1 dimensions. Holographic CFTs are *rare* among CFTs.
- Prototypical example in string theory: the 2d symmetric orbifold CFT of the D1D5 system recruited by Strominger-Vafa to compute S_{BH}.
- Slightly larger class: permutation orbifolds Haehl-Rangamani '14, Belin-Keller-Maloney '14.



- How did this prototypical holographic [S]CFT arise in string theory? Strominger-Vafa wrapped N_1 D1-branes on S^1 and N_5 D5-branes on $S^1 \times M_4$, where $M_4 = T^4$ or K3. When $\sqrt[4]{Vol(M_4)} \ll R(S^1)$, the physics of open strings ending on these wrapped D-branes becomes effectively 2d. Inspecting this system closely gave rise to AdS_3/CFT_2 duality.
- When the system is at nonzero temperature, giving rise to Hawking radiation, basic thermal physics considerations show that the lightest fluctuating modes must be fractionated. This is also needed for the open string/D-brane setup to correctly reproduce BH emission. The D1D5 boundstate is one multiply wound long string, rather than multiple singly wound short strings.

The D1D5 moduli space and the orbifold point

- The D1D5 system has a 20-parameter *moduli space* (continuous family of global minima).
- At one point in moduli space, it is described by black spacetime geometry: SUGRA BH.

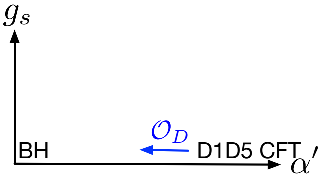


- At another, known as the orbifold point, it is described by a free 2d $\mathcal{N}=(4,4)$ SCFT with a symmetric product orbifold target space

$$(M_4)^N/S_N\,,$$

where $N \equiv N_1 N_5$, $c_{tot} = cN$. Theory on one M_4, seed SCFT, has $c = 4(1+\frac{1}{2})$.

- At the orbifold point, where the SCFT is free, it is believed to be dual to the tensionless limit of string theory on $AdS_3 \times S^3 \times M_4$.
- In this limit, CFT sparseness condition is not satisfied: α' corrections to SUGRA are large.



- Aim: to deform the D1D5 CFT towards the BH point, by [singlet] $\mathcal{O}_D$.

Conformal symmetry and Virasoro algebra

- The conformal group $SO(d,2)$ in d spacetime dimensions supplements the Lorentz group $SO(d-1,1)$ with 1 dilatation and $d-1$ conformal boosts

$$x^\mu \to \lambda x^\mu \,, \quad x^\mu \to \frac{x^\mu + a^\mu x^2}{1 + 2a_\nu x^\nu + a^2 x^2} \,.$$

- AdS_3/CFT_2 is qualitatively different than other AdS_{d+1}/CFT_d, because the symmetry algebra is enhanced to infinite-dimensional Virasoro.
- For a 2d CFT on $\mathbb{R} \times S^1$ with coordinates (t,σ), mapping from the cylinder to the plane via

$$z = e^{i(t+\sigma)} \,, \quad \bar{z} = e^{i(t-\sigma)} \,,$$

turns Fourier modes in (t,σ) into integer powers of z, $\bar{z}$. If we Wick rotate to 2d Euclidean space, $\bar{z} = e^{\tau - i\sigma}$ is the honest complex conjugate of $z = e^{\tau + i\sigma}$. Then we can merrily recruit bunches of useful facts from complex analysis. Time ordering on the cylinder becomes radial ordering in the plane.
- The stress tensor exists for any CFT. For 2d CFTs, the Virasoro algebra generators L_n are modes of the stress tensor: $T(z) = \sum_n L_n / z^{n+2}$. They obey

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0} \,, \quad n \in \mathbb{Z} \,.$$

OPE, primaries, and conformal anomaly

- In a CFT, there is a 1-1 correspondence between states and operators. The Operator Product Expansion, which is defined for QFTs, is especially useful for CFTs because its radius of convergence is finite. Write

$$\mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w}) \sim \sum_k \mathcal{C}_{ijk}(z-w)^{h_k-h_i-h_j}(\bar{z}-\bar{w})^{\bar{h}_k-\bar{h}_i-\bar{h}_j}\mathcal{O}_k(w,\bar{w})\,,$$

  where $\mathcal{C}_{ijk}$ are the structure constants of the CFT.
- Primary fields transform tensorially under conformal transformations, with weight h. Using the language of contour integrals, this can be rewritten as

$$T(z)\phi(w) = \frac{h\phi(w)}{(z-w)^2} + \frac{\partial_w\phi(w)}{(z-w)} + \textit{nonsingular}\,.$$

- The stress tensor is not a true tensor when $c \neq 0$,

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \textit{nonsingular}.$$

  The finite form of the transformation of T under $z \to f(z)$ is

$$T(z) \to [f'(z)]^2\, T(f(z)) + \frac{c}{12}\frac{[f'(z)f'''(z) - \frac{3}{2}(f''(z))^2]}{(f'(z))^2}\,.$$

D1D5 SCFT field content

- Our $(M_4)^N/S_N$ orbifold mods out by the symmetric group S_N. This is different from $\mathbb{Z}_n$ cyclic group orbifolds sometimes used in superstring model building.
- What are the symmetries and fields of our seed SCFT? Suppose $M_4 = T^4$.
  1. $SU(2)_L \times SU(2)_R$ R-symmetry,
  2. $SO(4)_I = SU(2)_1 \times SU(2)_2$ symmetry.
- Four real bosons, X^i $i \in \{1, \cdots, 4\}$. Write as doublets of $SU(2)_1$ and $SU(2)_2$:

$$X^{\dot{A}A} = \frac{1}{\sqrt{2}} X^i (\sigma^i)^{\dot{A}A}.$$

  Four real fermions in the left moving and four in the right moving sector. Combine to form complex fermions which are doublets of $SU(2)_L$ and $SU(2)_2$:

$$\psi^{\alpha\dot{A}},\ (\psi^{\alpha\dot{A}})^\dagger = -\epsilon_{\alpha\beta}\epsilon_{\dot{A}\dot{B}}\psi^{\beta\dot{B}}.$$

- Generators of superconformal algebra:-

$$\begin{aligned} T &= \frac{1}{4}\epsilon_{\dot{A}\dot{B}}\,\epsilon_{AB}\,\partial X^{\dot{A}A}\,\partial X^{\dot{B}B} + \frac{1}{2}\epsilon_{\alpha\beta}\,\epsilon_{\dot{A}\dot{B}}\,\psi^{\alpha\dot{A}}\partial\psi^{\beta\dot{B}}, \\ G^{\alpha A} &= \sqrt{2}\epsilon_{\dot{A}\dot{B}}\,\psi^{\alpha\dot{A}}\,\partial X^{\dot{B}A}, \\ J^a &= \frac{1}{4}\epsilon_{\dot{A}\dot{B}}\epsilon_{\alpha\beta}\,\psi^{\alpha\dot{A}}\,(\sigma^{*a})^{\beta}_{\gamma}\,\psi^{\gamma\dot{B}}. \end{aligned}$$

- Chiral primaries: $G^{+A}_{-\frac{1}{2}}|\chi\rangle = 0$, $h = m$, correspond to SUGRA modes in bulk. We are interested in anomalous dimensions of low-lying *string states*.

Sectors of symmetric orbifold CFT

- Lunin-Mathur '00 '01 developed a covering space method to analyze the physics of the $(M_4)^N/S_N$ orbifold CFT. Here are several pertinent details.
- The orbifold action mods out by S_N. This cuts some states from the spectrum. However, other new states appear – in the twisted sector.
- The untwisted sector contains S_N invariant sums and products over operators built from N copies of the seed theory. The twisted sector contains operators with boundary conditions that twist copies of the M_4 together.
- Twisted BCs are implemented by bare twist operators σ_n of length n with conformal weights $h = \bar{h} = \frac{1}{24}c(n - \frac{1}{n})$. They act on a field ϕ in the theory as

$$\sigma_n : \quad \phi_{(1)} \to \phi_{(2)} \to \ldots \to \phi_{(n)} \to \phi_{(1)} ,$$

  where the subscript (j) is a copy index. An S_N invariant is obtained by taking a normalized sum over the full orbit of the representative permutation.
- Aside: another context in which twist operators show up is in computing Rényi entropies in AdS/CFT by making use of the replica trick. This trick has been recruited to prove the Ryu-Takayanagi '06 formula, which geometrized the entanglement entropy of a region $\mathcal{A}$ in the CFT with its complement in terms of the area of the extremal surface in the bulk whose boundary is $\mathcal{A}$.

Twists, deformation, and fractional moding

- Twisted sector states matter physically: they dominate the density of states of the symmetric orbifold theory at high energy, where the growth is Hagedorn.
- Deformation operator $\mathcal{O}_D$ we use to perturb towards BH point in moduli space is a singlet under all $SU(2)$s, and is built as $\mathcal{O}_D \sim G_{-1/2}\bar{G}_{-1/2}\,\sigma_2$. This motivated us to investigate how OPEs of operators with twisty parts behave.
- Previously, in 1703.04744 with Burrington+Jardine, we were able to (a) uncover how the OPE of an untwisted operator and a twisted operator ($\mathcal{O}_D$) on the base descends from knowledge of the OPE on the cover, and (b) use it to discern how operator mixing happens for a low-lying string state.
- Bare twists are the lowest weight operators in the twisted sector. Their excitations by *fractional modes* fill out the rest of the twisted sector and depend on the details of the seed CFT. Fractionally moded operators

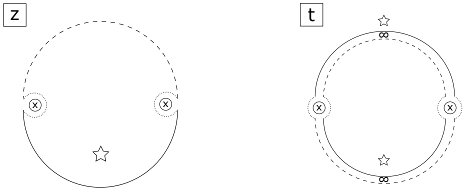
$$\phi_{-m/n} = \oint \frac{dz}{2\pi i} \sum_{k=1}^{n} \phi_{(k)}(z) e^{-2\pi i m(k-1)/n} z^{h-m/n-1}$$

  are only well defined in the presence of a length n twist operator. As $z \to z e^{2\pi i}$, the phases and the sum take care of the fractional power of z.
- Fractional modes of J are used to build twisted chiral primaries of D1D5 SCFT.

Map to the cover and ramification

- The magic of the Lunin-Mathur method is in the map. Twist operator insertions are ramified to the identity in the covering space. An example of a map that achieves this for twist operators at $z = 0$ and $z = 1$ is

$$z(t) = \frac{t^2}{2t - 1}$$



- Crucial to keep track of the conformal anomaly, by evaluating the Liouville action in transforming to the cover, carefully regulating all IR and UV ∞s.
- The large N behaviour of the m-point functions of bare twists is universal, depending on only three pieces of CFT data: c, N, and the twist lengths. For this reason, we conjecture that the twist-twist OPE in a general bosonic symmetric orbifold CFT should lift to a c-number operator in the cover, and that it should depend only on c, N, n_i. The Schwarzian depends only on c and the map, which is fixed fully by ramifications given by twist insertions.

Fractional Virasoro algebra

- We found is that it is possible to find a closed algebra for fractional Virasoro modes. This only makes sense within the twisted sector, but nonetheless it is very useful for our purposes.

$$\left[L_{k/n}, L_{k'/n}\right]|\sigma'\rangle \to \oint \frac{dt_2}{2\pi i}\oint_{t_1=t_2}\frac{dt_1}{2\pi i} z(t_1)^{k/n+1} z(t_2)^{k'/n+1}\left(\frac{dz}{dt_1}\right)^{-1}\left(\frac{dz}{dt_2}\right)^{-1} \left(T(t_1)-\frac{c}{12}\{z(t_1),t_1\}\right)\left(T(t_2)-\frac{c}{12}\{z(t_2),t_2\}\right)\sigma'_{\uparrow}\,.$$

- The $\oint dt_1$ around t_2 picks out the simple pole as $t_1 \to t_2$. Since t_2 is a non-ramified point, the Schwarzian does not blow up, and neither does the other part of the integrand. The only singularities in the above arise from the $T - T$ OPE on the cover. Straightforward contour integral algebra yields

$$[L_{k/n}, L_{k'/n}] = \frac{k-k'}{n} L_{(k+k')/n} + \frac{cn}{12}\left(\left(\frac{k}{n}\right)^2 - 1\right)\frac{k}{n}\,,$$

  which is the Virasoro algebra. Note that the central term only involves cn, the copies parallel to the cycle of the length-n twist. The most interesting thing about this result is that the answer is independent of the details of the map.

When is a fractional Virasoro primary?

- Consider the lowest-lying excitation of a bare twist of length n, $L_{-1/n}\sigma_n$. By lifting to the cover, we found that this is zero – which is not obvious from the perspective of the base space. We can more easily understand it by computing the norm from the fractional Virasoro algebra: it is indeed a null state.
- How about the cases $L_{-k/n}\sigma_n$? Are they primaries? Another short calculation from the fractional Virasoro algebra shows that they are, if $k \leq n+1$.
- Even though excitations with $k > n+1$ are not primary, it turns out that they can be combined with descendants to make new primaries. For example, $L_{-(n+2)/n}\sigma_n$ is not quasiprimary by itself, but the following linear combination

$$\left[1 - \frac{1}{2h-2}L_{-1}L_1\right] L_{-(n+2)/n}\sigma_n$$

  is annihilated by L_1. We found a related projection process in our earlier paper.
- We can generalize this story to any $k > n+1$, giving an infinite number of primaries. Further, we conjecture that these primary fractionally moded Virasoro operators can fully account for the exchanges in a four point function.

Coincidence limit of four-point function

- One way to check which operators appear in the OPE of two twist operators, say σ_2 and σ_n, is to compute the four-point correlation function

$$\frac{\langle \sigma_n(0,0)\sigma_2(1,1)\sigma_2(w,\bar{w})\sigma_n(\infty,\bar{\infty})\rangle}{\langle \sigma_2(0,0)\sigma_2(1,1)\rangle\langle \sigma_n(0,0)\sigma_n(\infty,\bar{\infty})\rangle}$$

  and study the coincidence limit $(w,\bar{w}) \to (0,0)$. The reason why we divided out by a product of two-point functions is to set the normalization correctly.
- There are three options for what might happen: no cycle overlaps, one overlap (fusing to σ_{n+1}), or two overlaps (σ_{n-1}). The full S_N invariant answer is a sum of disconnected, $g = 0$, and $g = 1$ pieces, where g is cover genus. Physically, we do not expect which operators show up in the OPE to depend on g.
- By recruiting the (also carefully normalized) OPE for two twists, and plugging it into the four-point correlator above, we find

$$\sum_k |w|^{-2h_2-2h_n} w^{h_k} \bar{w}^{\bar{h}_k} C_{2nk}^2 \,,$$

  where the three-point function is normalized properly to return the structure constant. Therefore, by comparing with the actual four-point function of bare twists, we can extract data on the spectrum of $(h_k, \bar{h}_k, C_{2nk})$.

Order by order results

- We were not able to get information about all operators exchanged at arbitrary order, but we were able to check at several nontrivial orders to see the lowest-lying contributors, enabling us to test how well our conjecture works.
- At first non-leading order, there is no term corresponding to an excitation of $1/(n+1)$, which agrees with our earlier finding that $L_{-1/n}\,\sigma_n = 0$.
- At second non-leading order, after chasing the conformal anomaly down, and finding that the Schwarzian is where the main action is, we find that indeed, $L_{-2/(n+1)}\,\sigma_{n+1}$ fully accounts for the exchange.
- At third non-leading order, again it is the Schwarzian that is key, and after dealing with a normalization subtlety we also find success: $L_{-3/(n+1)}\,\sigma_{n+1}$ fully accounts for the exchange.
- How about operators with $k > n+1$? Here the computations become increasingly arduous, because of the necessary projection procedure to make a primary, and the fact that descendants of lower weight operators will also contribute in the coincidence limit. Also, the number of possible operators grows. So we checked the simplest nontrivial case with $n=2$, $L_{-5/3}\,\sigma_3$. Interestingly, here the primary does not contribute to the four-point function; the descendant accounts for all of it, and the primary is not a null state.

Remarks

- From the point of view of the covering space, these results are not all that surprising. When we lift bare twists to the cover, they are replaced by the identity. Then the OPE of the identity should contain the identity, plus all descendants under the cover Virasoro, which are the cover representatives of fractional Virasoro excitations. This perspective also explains the vanishing of $L_{-1/n}|\sigma_n\rangle$: it would roughly lift to $L_{-1}|0\rangle$, which vanishes.
- Can we find the OPE using more algebraic techniques? In a CFT, if we know the coefficient of a primary ϕ_p appearing in the OPE of two other primaries ϕ_1 and ϕ_2, then the descendants of ϕ_p must also appear with coefficients given by conformal invariance. The coefficients are read off by applying an arbitrary L_n operator to both sides of the equation and obtaining recurrence relations. These can be solved iteratively to find the the coefficients level by level.
- But in our case, there are obstructions to standard contour pull tricks that block this method: singularities in the Schwarzian, and branch points in the fractional powers in the integrand. The three point function also suffers from contour pull failures, originating in fractional mode descendants. Luckily, in exploring exchange channels in four-point functions, these conveniently cancel.
- For general info, must resort to other techniques, like working on the cover.

Summary and future directions

- In the context of deforming D1D5 SCFT towards BH spacetime, we explored OPE of twist operators of generic bosonic $(M_4)^N/S_N$ orbifold CFT at large N.
- We conjectured that at large N the OPE of bare twist operators contains only bare twists and excitations of bare twists with *fractional Virasoro modes*. These fractionally excited operators are the only ones that depend exclusively on N, the seed theory central charge, and the lengths of the twists, agreeing with the general structure of correlators of bare twists found in the literature.
- To provide evidence for this, we studied the coincidence limit of a four point function of bare twist operators to several non-leading orders. We showed how the coefficients of these powers can be reproduced by considering bare twist operators excited by fractional Virasoro modes in the exchange channels.
- Work currently in progress also with Thomas de Beer (PhD student):-
  1. Investigate the SCFT case with fermions. Expect contributions not only from fractional Virasoro modes but also from SUSY and R-currents. Spin fields will appear in even the simplest bare twist of length 2.
  2. Can we use twist-twist OPE knowledge to constrain maps to the covering space pertinent to four-point correlators, which are not known in closed form? How far along this path can we push?