

11D supergravity in 4D, $\mathcal{N} = 1$ language

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Based on several papers with K. Becker, M. Becker, D. Butter, S. Guha, W. Linch, and S. Randall leading up to [1709.07024](#).



“Now, think of a cheetah that has been captured and thrown into a miserable cage in a zoo. It has lost its original grace and beauty, and is put on display for our amusement. We see only the broken spirit of the cheetah in the cage, not its original power and elegance. The cheetah can be compared to the laws of physics, which are beautiful in their natural setting. The natural habitat of the laws of physics is higher-dimensional space-time. However, we can only measure the laws of physics when they have been broken and placed on display in a cage, which is our three-dimensional laboratory. We can only see the cheetah when its grace and beauty have been stripped away.”

– Peter G. O. Freund

11D SUGRA – The simplest supergravity

Eleven is the highest dimension in which you can have supergravity, and this theory has many nice properties.

- It's the low energy limit of M-theory.
- Field content is uniquely fixed, g_{MN} , C_{MNP} , ψ_M^A .
- Action is uniquely fixed

$$S_{11D} = \frac{1}{2\kappa^2} \int d^{11}x \sqrt{-g} \left(R - \frac{1}{2} |dC|^2 \right) - \frac{1}{12\kappa^2} \int C \wedge dC \wedge dC.$$

There are 32 real supercharges (one Majorana spinor).
SUSY closes only on-shell.

4D $\mathcal{N} = 1$ SUGRA – The other simplest supergravity

Can get this supergravity in many ways, e.g. low-energy limit of M-theory on a G_2 -holonomy manifold.

- 4 real supercharges (one Weyl spinor)
- The field content is not unique, but comes in supermultiplets, e.g.
 - chiral (two real scalars, one spin 1/2),
 - vector (one vector potential, one spin 1/2),
 - gravity (one spin 2, one spin 3/2).
- The action is not unique, but is determined by some functions $K(\Phi, \bar{\Phi})$, $W(\Phi)$, $h_{IJ}(\Phi)$.

Off-shell superspace

A very nice aspect of working with 4D $\mathcal{N} = 1$ supersymmetry is the existence of a simple off-shell superspace.

- Can introduce auxiliary fields so SUSY closes off-shell.
- Then we have off-shell superspace $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$,
- Supersymmetries act as differential operators

$$Q_\alpha, \bar{Q}_{\dot{\alpha}}, \quad \mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}.$$

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- Supersymmetries act as differential operators

$$Q_\alpha, \bar{Q}_{\dot{\alpha}}, \quad \mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}.$$

- The action can be written

$$\int d^4x d^4\theta E K(\Phi, \bar{\Phi}) + \left(\int d^4x d^2\theta \mathcal{E} \left(W(\Phi) + h_{IJ}(\Phi) W^{I\alpha} W_\alpha^J \right) + \text{c.c.} \right),$$

where $W_\alpha^I = -\frac{1}{4} \bar{\mathcal{D}}^2 \mathcal{D}_\alpha V^I$.

Advantages to going off-shell

Having a supersymmetry algebra that closes off-shell, and the associated superspace, is very useful.

- The supersymmetry transformations do not get corrections; corrections are sequestered in the action, and can be classified/enumerated more easily.
- It's a good organizational principle for things like non-renormalization theorems.

Goal

We would like the advantages of an off-shell formulation in the context of 11D supergravity, but with a superalgebra that large, it just can't be done.

Motivated in particular by the example of working around a background $\mathbb{R}^4 \times Y$, with Y being a G_2 manifold, we can try a different approach.

Goal

We would like the advantages of an off-shell formulation in the context of 11D supergravity, but with a superalgebra that large, it just can't be done.

Motivated in particular by the example of working around a background $\mathbb{R}^4 \times Y$, with Y being a G_2 manifold, we can try a different approach.

- Consider an $11 = 4 + 7$ split of the coordinates. The 32 supercharges become $\mathcal{N} = 8$ supersymmetry in 4D.
- Forget about 7 of the 8 supersymmetries (for instance in the G_2 case, 7 of the 8 are broken by the background).
- Find an off-shell formulation of the remaining 4D $\mathcal{N} = 1$.

Formally, the result can be thought of as a theory on $\mathbb{R}^{4|4} \times Y$.

Kaluza-Klein revisited

In our 4 + 7 split, we will write

$$g_{MN} = \begin{pmatrix} g_{mn} + \gamma_{kl} \mathcal{A}_m^k \mathcal{A}_n^l & \gamma_{ik} \mathcal{A}_n^k \\ \gamma_{jk} \mathcal{A}_m^k & \gamma_{ij} \end{pmatrix},$$

$$C_{MNP} \rightarrow C_{ijk}, C_{mij}, C_{mni}, C_{mnp},$$

$$\psi_M^A \rightarrow \psi_m^{\alpha I} = \psi_m^\alpha \eta^I + i \psi_{mj}^\alpha (\Gamma^j)^{IJ} \bar{\eta}_J, \quad \psi_i^{\alpha I} = \psi_i^\alpha \eta^I + i \psi_{ij}^\alpha (\Gamma^j)^{IJ} \bar{\eta}_J,$$

where η^I is a fixed complex spinor that picks out the $\mathcal{N} = 1$. Similarly we can decompose the diffeomorphism, gauge, and local SUSY parameters $\xi^M, \Lambda_{MN}, \epsilon^A$.

The three-form

Let's start with the three-form and try to embed everything into superfields, starting just with rigid superspace.

$$\begin{aligned} C_{ijk} &\in \Phi_{ijk}, & \bar{D}_{\dot{\alpha}} \Phi &= 0, \\ C_{mij} &\in V_{ij}, & V &= \bar{V}, \\ C_{mni} &\in \Sigma_{i\alpha}, & \bar{D}_{\dot{\alpha}} \Sigma_{\alpha} &= 0, \\ C_{mnp} &\in X, & X &= \bar{X}. \end{aligned}$$

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The first two are the familiar chiral and vector superfields, valued in internal three-forms and two-forms respectively. Σ_{α} is a chiral spinor superfield valued in internal one-forms, and X is a real scalar superfield.

Abelian gauge transformations

It's also not too hard to work out how to fit the three-form gauge transformations (Λ_{MN}) into this superfield language. If we use the symbol d_Y to denote the exterior derivative on Y , we have

$$\Lambda_{ij} \in \Lambda_{ij}, \quad \Lambda_{mi} \in U_i, \quad \Lambda_{mn} \in \Upsilon_\alpha.$$

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and

$$\begin{aligned}\delta\Phi &= d_Y\Lambda, \\ \delta V &= \frac{1}{2i} (\Lambda - \bar{\Lambda}) - d_Y U, \\ \delta\Sigma_\alpha &= -\frac{1}{4} \bar{D}^2 D_\alpha U + d_Y \Upsilon_\alpha, \\ \delta X &= \frac{1}{2i} (D^\alpha \Upsilon_\alpha - \bar{D}_{\dot{\alpha}} \bar{\Upsilon}^{\dot{\alpha}}).\end{aligned}$$

Field strengths and Bianchis

We can easily build invariant field strengths,

$$E = d_Y \Phi,$$

$$F = \frac{1}{2i} (\Phi - \bar{\Phi}) - d_Y V,$$

$$W_\alpha = -\frac{1}{4} \bar{D}^2 D_\alpha V + d_Y \Sigma_\alpha,$$

$$H = \frac{1}{2i} (D^\alpha \Sigma_\alpha - \bar{D}_{\dot{\alpha}} \bar{\Sigma}^{\dot{\alpha}}) - d_Y X,$$

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These obey Bianchi identities

$$\begin{aligned}0 &= d_Y E, \\0 &= \frac{1}{2i} (E - \bar{E}) - d_Y F, \\0 &= -\frac{1}{4} \bar{D}^2 D_\alpha F + d_Y W_\alpha, \\0 &= \frac{1}{2i} (D^\alpha W_\alpha - \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}) - d_Y H, \\0 &= -\frac{1}{4} \bar{D}^2 H + d_Y G.\end{aligned}$$

Chern-Simons term

And the symmetry structure uniquely fixes a gauge invariant Chern-Simons action

$$S_{CS} = \frac{1}{\kappa^2} \int d^4x d^7y L_{CS},$$

$$\begin{aligned} L_{CS} = & -\frac{i}{12} \int d^2\theta \left[\Phi \left(EG + \frac{1}{2} W^\alpha W_\alpha - \frac{i}{4} \bar{D}^2 (FH) \right) \right. \\ & \left. + \Sigma^\alpha \left(EW_\alpha - \frac{i}{4} \bar{D}^2 (FD_\alpha F) \right) \right] \\ & - \frac{1}{12} \int d^4\theta \left[V \left(\frac{1}{2} (E + \bar{E}) H + \omega(W, F) \right) - X \left(\frac{1}{2} (E + \bar{E}) F \right) \right] \\ & + h.c. \end{aligned}$$

where $\omega(W, F)$ is a Chern-Simons superfield.

Including \mathcal{A}_m^i

It's not difficult to also include \mathcal{A}_m^i and the internal diffeomorphisms, represented as a non-abelian vector superfield with accompanying field strength \mathcal{W}_α^i .

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- The effect is to covariantize the superderivatives, $D_\alpha \rightarrow \mathcal{D}_\alpha$.
- All fields are charged under this symmetry, but we can construct covariant field strengths for the tensor hierarchy above, and again there is a uniquely fixed Chern-Simons action.
- We first worked this structure out by brute force, essentially writing down all possible terms in the action and fixing coefficients by gauge invariance, but retrospectively we understood how to get everything using the technology of superforms.

Taking inventory

Unfortunately, we have too many dynamical fields.

- The 35 Φ_{ijk} contain all the C_{ijk} , but what are the partner scalars?
- There are 7 real scalars in the $\Sigma_{i\alpha}$.
- There is a complex scalar in X .

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Unaccounted for are the 28 scalars that should be coming from the internal metric.

In the case where we truncate M-theory on G_2 to massless modes, the answer was known, the modes of the metric can be rewritten as a (harmonic) G_2 -structure three-form φ_{ijk} , and this paired up with C_{ijk} .

In our case there is still a mismatch of 7 (these can never be harmonic, so no contradiction) plus the 9 from Σ and X .

Matching component actions

A little bit more was known from the massless truncation, namely that the Kähler potential was essentially the volume, where we view the metric as being built out of the G_2 -structure three-form φ , which is the lowest component of the superfield strength F ,

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So there's an exercise we can do, even though there are these mismatches in field content. We can add the Kähler term above (actually it turns out that we need to multiply by $(G\bar{G})^{1/3}$ and expand around a background $G = 1$ to match everything), and integrate out auxiliary fields to get a component action.

Amazingly, a large number of coefficients match. We get all the “potential” terms of the component action, as well as kinetic terms for everything that isn't in the 7 representation of G_2 .

Gravitino multiplets

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Following Gates and Siegel, we can embed our spin- $3/2$ component fields in unconstrained spinor superfields $\Psi_{i\alpha}$ with a *large* gauge symmetry,

$$\delta\Psi_{i\alpha} = \Xi_{i\alpha} + D_\alpha\Omega_i,$$

where Ξ_i are chiral spinor superfields and Ω_i are complex scalar superfields.

Spin two multiplets

Similarly, we must introduce the gravity multiplet, which can be packaged in a superfield $H_{\alpha\dot{\alpha}}$, with

$$\delta H_{\alpha\dot{\alpha}} = \bar{D}_{\dot{\alpha}} L_{\alpha} - D_{\alpha} \bar{L}_{\dot{\alpha}}.$$

The gauge parameter L_{α} contains local 4D superconformal transformations. This is a standard presentation of old minimal supergravity, provided we have a compensator field to break the conformal symmetry.

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It turns out that G is just right to act as the compensator if

$$\delta_L X = D^{\alpha} L_{\alpha} + \bar{D}_{\dot{\alpha}} \bar{L}^{\dot{\alpha}}.$$

Consistency

We also need for consistency

$$\delta_L \Psi_{i\alpha} = 2i\partial_j L_\alpha,$$

and

$$\delta_{\Xi} \Sigma_{i\alpha} = -\bar{\Xi}_{i\alpha},$$

along with transformations of some fields under Ω_i , in particular

$$\delta_\Omega \mathcal{V}^i = -\frac{1}{2} (\Omega_i + \bar{\Omega}_i).$$

Quadratic action

Schematically,

$$S \sim S_{CS} - 3 \int d^4\theta \sqrt{g(F)} (G\bar{G})^{\frac{1}{3}} \left(1 - \frac{1}{4} (G\bar{G})^{-\frac{2}{3}} g^{ij} H_i H_j + \dots \right)$$

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We build the quadratic action by expanding around a fixed background given by φ_{ijk} , $G = 1$ (i.e. $X = \theta^2$).

$$S = \frac{1}{\kappa^2} \int d^4x d^7y \left[\int d^4\theta L_D + \left(\int d^2\theta L_F + h.c. \right) \right].$$

F-term piece

$$L_F = -\frac{i}{288} \epsilon^{ijklmnp} \Phi_{ijk} \partial_l \Phi_{mnp} + \frac{1}{24} \tilde{\varphi}^{ijkl} G \partial_i \Phi_{jkl} \\ + \frac{1}{32} \tilde{\varphi}^{ijkl} W_{ij}^\alpha W_{\alpha kl} + \frac{1}{4} g_{ij} W^{\alpha i} W_\alpha^j$$

In particular, this contains the G_2 superpotential $W(\Phi) \sim \int_Y \Phi \wedge d_Y \Phi$, as well as being consistent with the expected kinetic couplings $h^{j,k\ell}(\Phi) W_{ij}^\alpha W_{k\ell\alpha} \sim \int_Y \Phi \wedge W^\alpha \wedge W_\alpha$.

D-term piece

$$\begin{aligned}
 L_D = & -H^a \square H_a + \frac{1}{8} D^2 H_a \bar{D}^2 H^a - (\partial_a H^a)^2 + \frac{1}{48} ([D_\alpha, \bar{D}_{\dot{\alpha}}] H^{\alpha\dot{\alpha}})^2 \\
 & - \frac{1}{3} \bar{G} G + \frac{2i}{3} (G - \bar{G}) \partial_a H^a \\
 & - \frac{1}{18} (G + \bar{G} - \frac{1}{2} [D_\alpha, \bar{D}_{\dot{\alpha}}] H^{\alpha\dot{\alpha}}) \varphi^{ijk} F_{ijk} \\
 & - \frac{1}{9} F_{1ijk}^2 - \frac{1}{12} F_{7ijk}^2 + \frac{1}{12} F_{27ijk}^2 \\
 & - \frac{1}{2} \left[\partial_i H_{\alpha\dot{\alpha}} - \frac{1}{2i} (\bar{D}_{\dot{\alpha}} \Psi_{\alpha i} + D_\alpha \bar{\Psi}_{\dot{\alpha} i}) \right]^2 \\
 & - \frac{1}{4} \left[H_i + \frac{1}{2i} (D^\alpha \Psi_{\alpha i} - \bar{D}_{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha} i}) \right]^2 \\
 & + \frac{1}{2} \left\{ \Psi_i^\alpha \left[W_\alpha^i - \frac{i}{2} \varphi^{ijk} (\partial_j \Psi_{\alpha k} + W_{\alpha jk}) - \frac{1}{12} \tilde{\varphi}^{ijkl} D_\alpha F_{jkl} \right] + \text{h.c.} \right\} \\
 & + \dots
 \end{aligned}$$

Mission Accomplished!

Fixing gauges and integrating out all auxiliary fields exactly reproduces the quadratic component action for M-theory expanded around a G_2 compactification!

Future directions

- It would be quite satisfying to extend this to full nonlinear order. Some of my collaborators have recently posted a paper (1803.00050) in that direction, working to all orders in everything except $\Psi_{i\alpha}, H_{\alpha\dot{\alpha}}$.
- Even short of that, providing linearized expansions around Freund-Rubin or other solutions of interest would be worthwhile.
- We would like to apply these results and techniques to finally study higher derivative corrections to supergravity.
- We can repeat the story in some other contexts, e.g. heterotic, type II, Spin(7), etc.

Thanks!