CHERN-SIMONS THEORY AND LINK INVARIANTS

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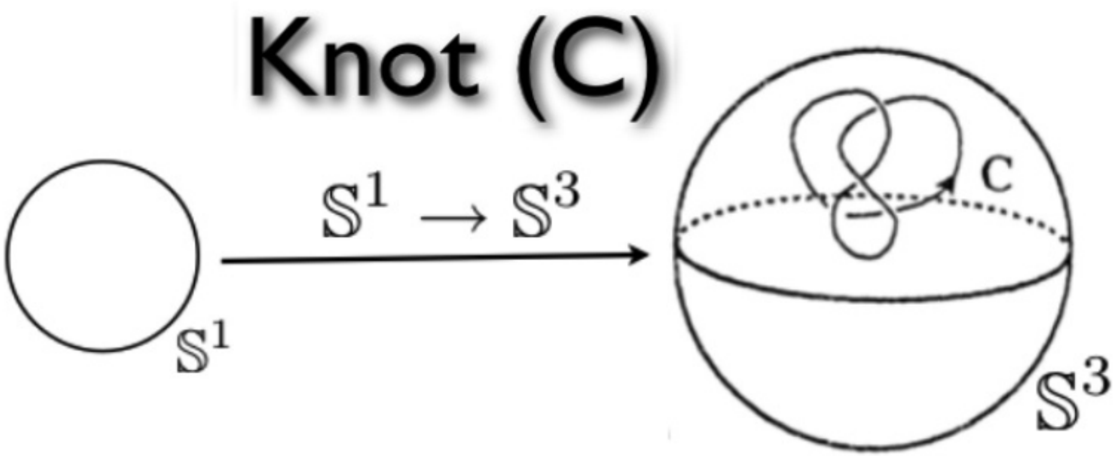
Content

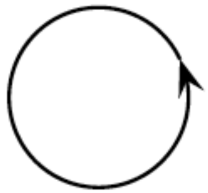
- Introduction to Knot, Link and Jones Polynomial
- Surgery theory
- Axioms of Topological Quantum Field Theory
- Wilson loops' expectation value

Knot and Link?

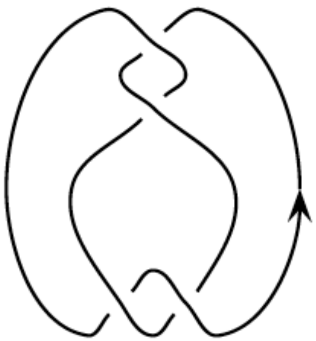
- ***Knot*** is a smooth non-intersecting closed curved in three-manifold. Oriented closed curves are oriented knots.



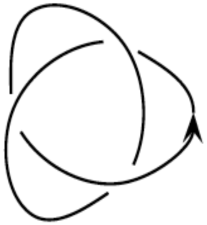
- With given knot, we associate a ***knot diagram*** obtained by projecting the knot on to a plane.



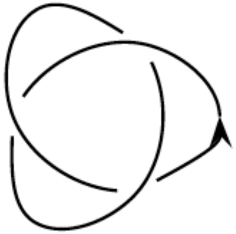
The *trivial knot*



The *figure-eight knot*

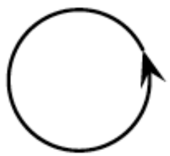


The *right-handed trefoil knot*



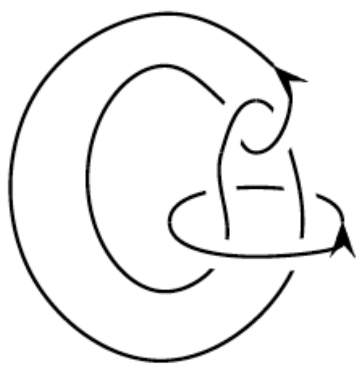
The *left-handed trefoil knot*

- ***Link*** is a collection of non-intersect loops (knots). A knot then is a single component link, we also have ***link diagram*** as the projection of a link on plane.

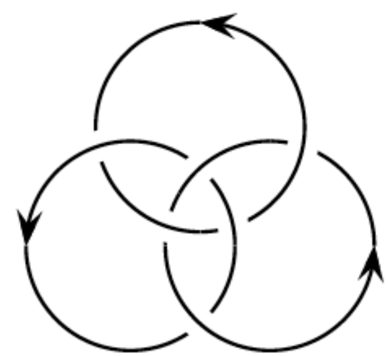


The *trivial 3-component link*

The *Hopf link*

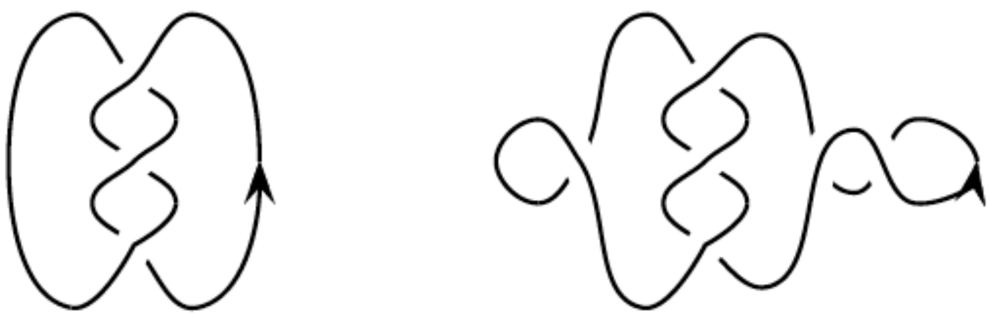


The *Whitehead link*



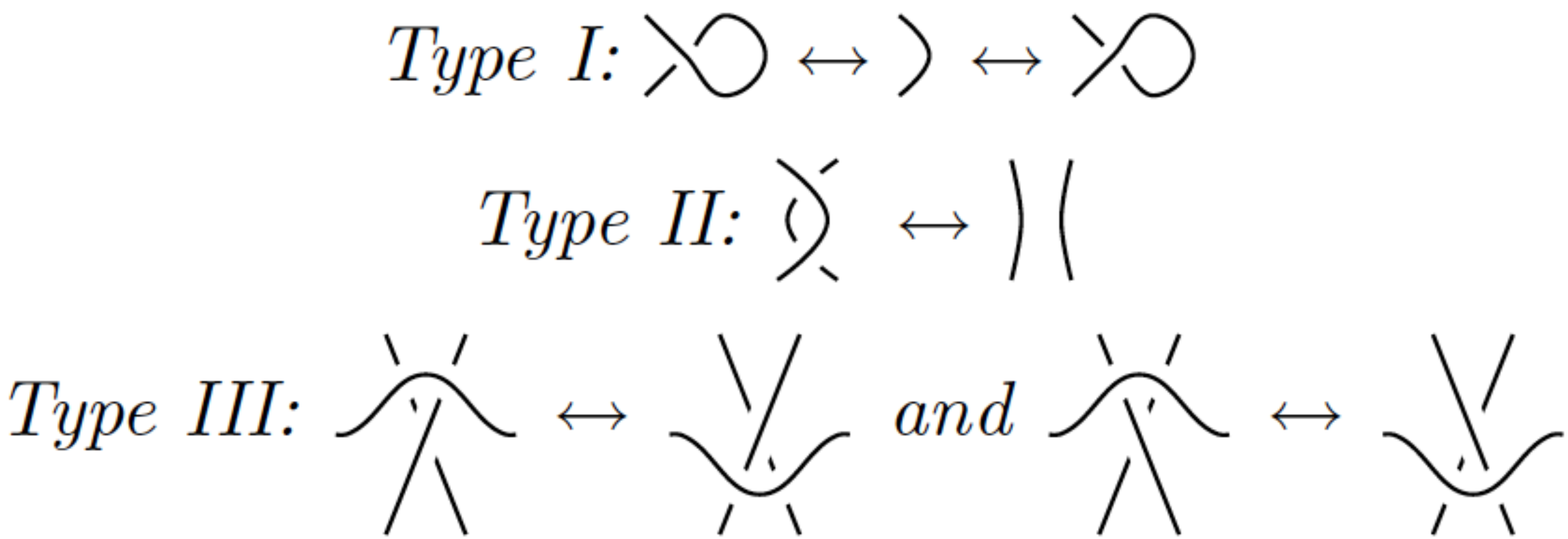
The *Borromean rings*

- **Remarks**: A link is well-determined by one of its diagram.
- One knot (link) can have different knot (link) diagrams depending on how we map the knot on the plane.
- We have following diagram for right-hand trefoil knot

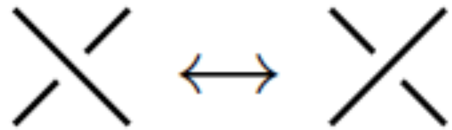


- **Reidemeister theorem (1926):**

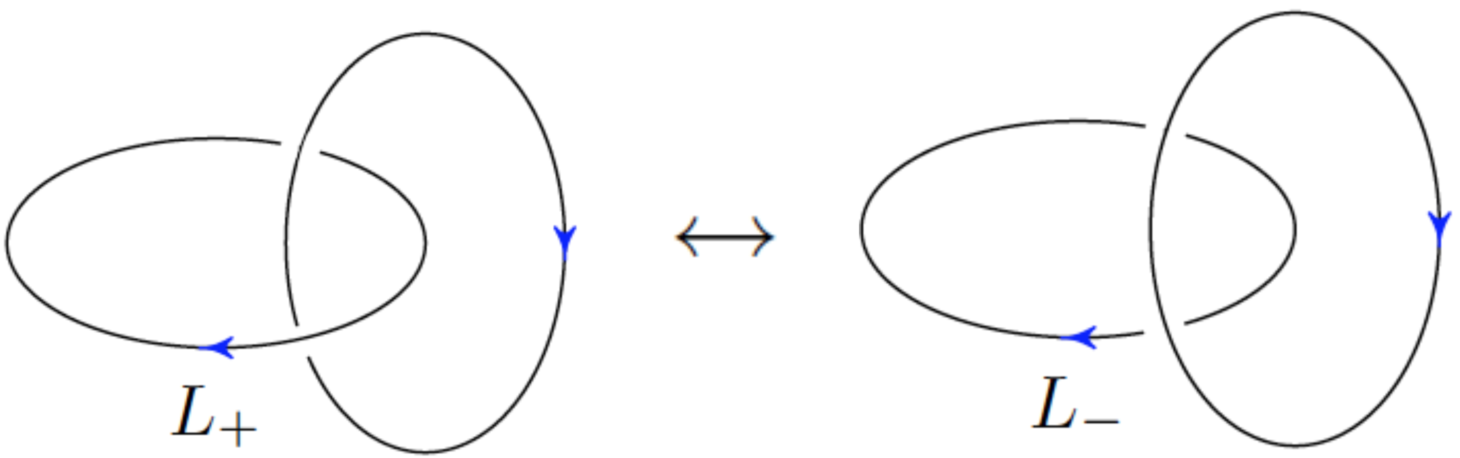
Up to orientation preserving diffeomorphism of the plane, **two** diagrams of **a** link can be related by a **finite** sequence of (local) **Reidemeister moves**



- ***Crossing change*** is a local modification of the type



- Note that it is not a Reidemeister move.
- ***Theorem***: Any link can be unknotted by a finite number of crossing changes.



Link invariant

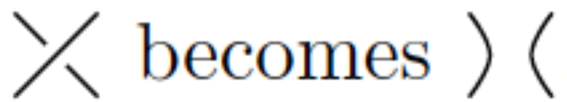
- *Link invariant* is a map from **set of links** to and **other set** (In the case of Jones polynomial the **other set are polynomials)** .
- Such a map can be defined as function of diagrams that is invariant under Reidemeister moves.

- *Positive crossing* in a diagram is the crossing 

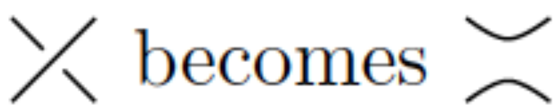
- *Negative crossing* in a diagram is the crossing 

Jones Polynomial and link invariant

- Let D be an unoriented link diagram. A crossing of D can be removed in two different ways.
- Left hand (L):



- Right hand (R):

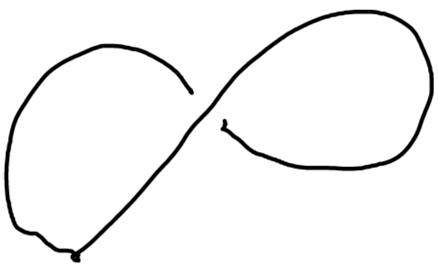


- Let C(D) be the set of crossing of D and f be a map from C(D) to $\{L, R\}$, then D_f will denote the diagram by remove crossing *x* in left-hand way if $f(x) = L$ and in the right-hand way if $f(x) = R$.
- D_f Is nothing but a collection of $n(D_f)$ circles embedded in the plane.
- We define ***Kauffman bracket*** <D> as

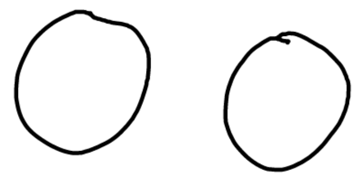
$$< D >= \sum_{f:C(D)\longrightarrow\{L,R\}} A^{(\sharp f^{-1}(L)-\sharp f^{-1}(R))}\delta^{(n(D_f)-1)}$$

- With $\delta = -A^2 - A^{-2}$
- Where A is a variable.

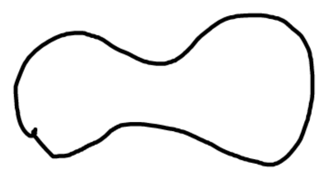
- Example



$\xrightarrow{f_1}$ L



$\xrightarrow{f_2}$ R



$n(D_{f_1}) = 2$

$$A \cdot \delta^{(2-1)}$$

$+$

$n(D_{f_2}) = 1$

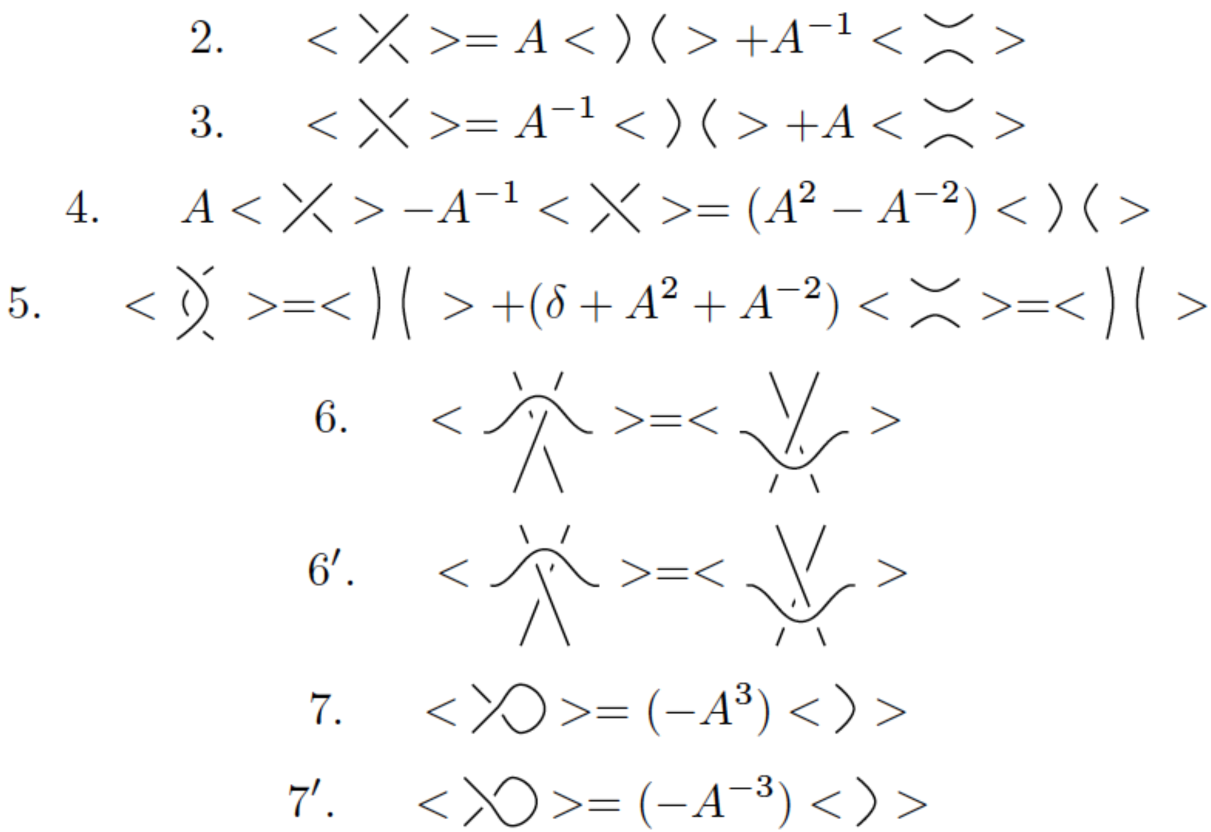
$$A^{-1} \delta^{(1-1)}$$

$$= -A^3 - A^{-1} + A^{-1} = -A^3$$

- Properties of Kauffman bracket

$$1. \qquad < \text{n disjoint circles} > = \delta^{n-1}$$

- In the following properties, the related brackets of diagrams are identical everywhere except where they are drawn



- *Define* $w(D)$ of an oriented link diagram D is the number of positive crossing minus the number of negative ones.
- **Jones polynomial** V(L) of an oriented link L is the polynomial from an oriented diagram D of L by

$$V(L) = (-A)^{-3w(D)} < D >_{A^{-2}=t^{1/2}}$$

- V is invariant of oriented links. It is unique invariant of oriented links that satisfies

1. *V(trivial knot)=1,*
2. *and the* skein relation:

$$t^{-1}V(\quad) - tV(\quad) = (t^{1/2} - t^{-1/2})V(\quad)$$

- We can prove (2) from property 4th of Kauffman bracket.

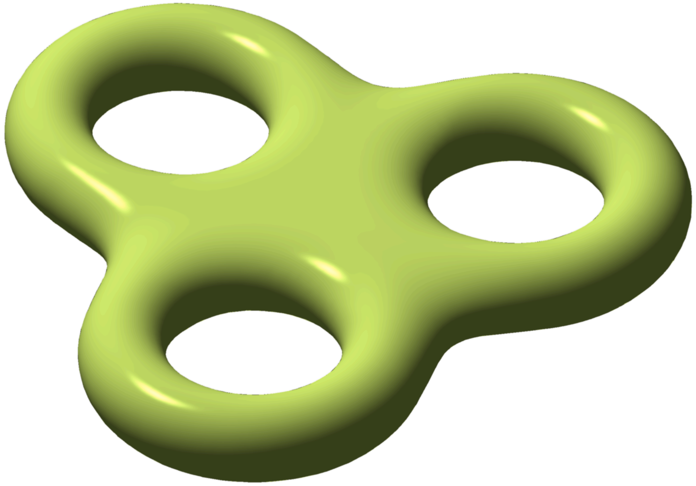
- **Some important remarks on Jones polynomial:**
- When we specify t, the value V(L) of each oriented link is unique. Jones polynomial is Link invriant.
- From the skien relations, and previous theorem, we can calculate Jones polynomial for any link with enough patient.
- The idea is any link can be unknotted by finite crossing changes, and the V(L) of a n-component trivial link is trivial

$$V(\text{trivial n-component link}) = (-t^{1/2} - t^{-1/2})^{n-1}$$

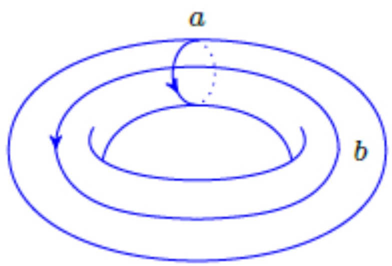
Surgery theory in 3-malnifolds

- **Heegard splitting and gluing:**
- Every compact orientable 3-manifold can be presented as gluing 2 handle-bodies with the same genus along their **common boundaries**.
- Gluing is merging 2 manifolds with boundary with a diffeomorphism map between their boundaries (we will see in the examples).

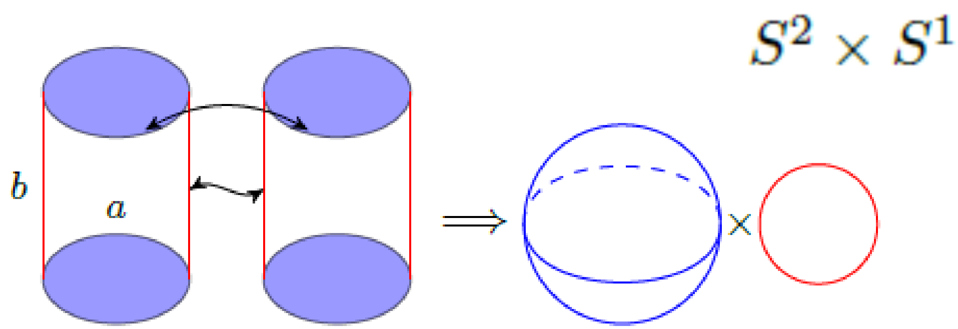
$$f : \partial \mathrm{M}_1 \to \partial M_2$$



- Let's see the **gluing** of 2 solid tori

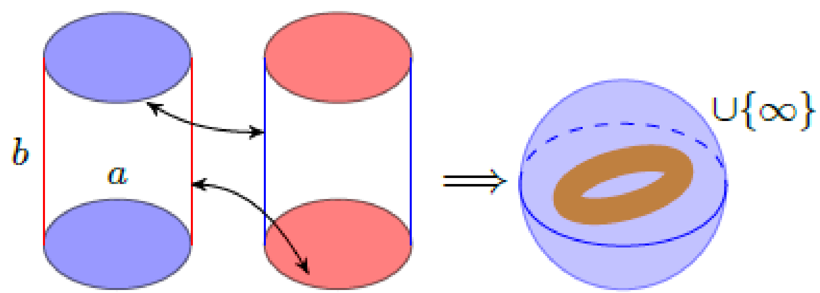


The smaller circle is the contractible a-cycle, and the meridian is the b cycle

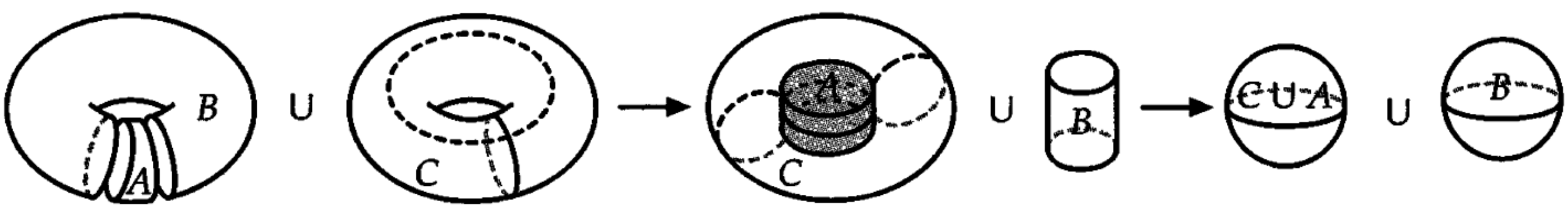


Gluing two solid tori, a cycle to a-cycle and b-cycle to b-cycle.

- If we gluing 2 tori after S-dual when we map contractible circle of one torus to the other meridian circle.



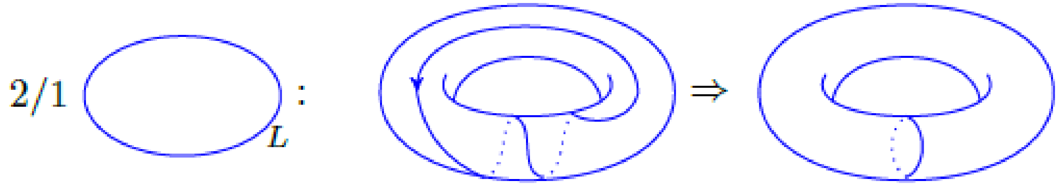
- Let’s see how does it happen



Gluing two solid tori together in this way yields S^3

- We can glue 2 D^3 on their boundaries to get S^3

- The most important surgeries are surgery along a link or surgery along a knot.
- We remove the tubular neighborhood of link L from S^3 and reglue it back after a diffeomorphism.
- The p/q (with p and q are coprime) surgery is the surgery with the specific diffeomorphism that pa+qb circle becomes contractible circle (a circle)



A 2/1 surgery along a link

- The surgery a long a knot which is the group of mapping torus to it self (SL(2,Z) group) can be decomposed in to S transformation and T transformation, the matrix form of the transformations are

$$T = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}, \quad S = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$

- And we also have

$$T^p = \begin{vmatrix} 1 & p \\ 0 & 1 \end{vmatrix}$$

- Which map pa+b circle into b circle. It is also called Dehn twist, we open the solid torus along a-circle and reglue after p twists.
- And S transformation exchanges the one-circles of the torus.
  - ***Why we care about surgery theory?***

Chern-Simons theory as TFT

- Consider C-S action

$$S = \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

- The C-S action is metric independent
- A is the **G**-gauge connection on principle bundle on base space **M.**
- k is integer named level (inverse of coupling constant)
- Integrate out A_t

$$\mathsf{F} = \mathsf{dA} + \mathsf{AA} = 0$$

(we have flat gauge curvature)

- Chern-Simons theory is an example of topological quantum field theory.
- TQFT has set of *operators* (observables) whose correlation functions do not depend on metric of base space **M.**

$$\frac{\delta}{\delta g_{\mu\nu}}\langle \mathcal{O}_{i_1}\cdots\mathcal{O}_{i_n}\rangle = 0$$

- In Chern-Simons theory, the operators are Wilson loop operators

$$W_R^{\mathcal{K}}(A) = \mathrm{Tr}_R U_{\mathcal{K}}, \qquad U_{\mathcal{K}} = \mathrm{P}\,\exp\oint_{\mathcal{K}} A$$

- It is the Wilson loop along oriented knot **K** in representation **R** of group **G.**

- Consider link **L** with L knot component, we can in principle calculate normalized correlation function

$$W_{R_1 \cdots R_L}(\mathcal{L}) = \langle W_{R_1}^{\mathcal{K}_1} \cdots W_{R_L}^{\mathcal{K}_L} \rangle = \frac{1}{Z(M)} \int [\mathcal{D}A] \Big(\prod_{\alpha=1}^{L} W_{R_\alpha}^{\mathcal{K}_\alpha} \Big) \mathrm{e}^{iS}$$

- The complicated link can be derive from simple knots by using surgery.

Axioms of TQFT (specify for C-S)

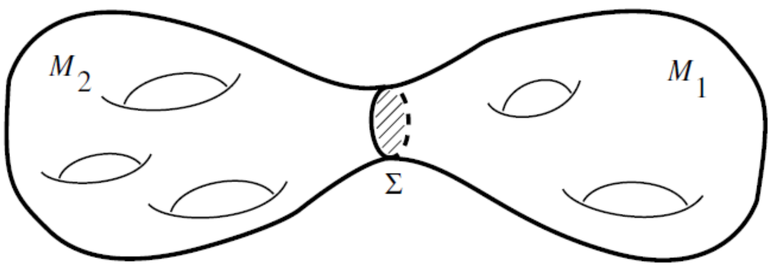
- Let M be a 3-manifold with boundary given by Riemann surface Σ , we can insert operator $\mathcal{O}$ as product of Wilson loops.
- We consider the case that Wilson loops do not intersect boundary of M.
- The path integral over M gives wave function is a function of value of field on Σ

$$\Psi_{M,\mathcal{O}}(\mathcal{A}) = \langle \mathcal{A} | \Psi_{M,\mathcal{O}} \rangle = \int_{A|_\Sigma = \mathcal{A}} \mathcal{D}A \, \mathrm{e}^{\mathrm{i}S} \, \mathcal{O}$$

- Associated to Riemann surface Σ we have Hilbert surface $\mathcal{H}(\Sigma)$ which obtain by canonical quantization of C-S theory on $\Sigma \times \mathbf{R}$ (**Hilbert space axiom**)

- We can consider two 3-manifolds M_1 and M_2 sharing common boundary Σ. We glue M_1 and M_2 by homeomorphism $f : \Sigma \to \Sigma$

$$M = M_1 \cup_f M_2$$



- Path integral over M_1 is state $|\Psi_{M_1}\rangle$

- In gluing process M_2 share the same boundary Σ but with opposite orientation, so it Hilbert space is the dual space $\mathcal{H}^*(\Sigma)$ (**Orientation axiom**) . The path integral over M_2 gives us a state $\langle\Psi_{M_2}| \in \mathcal{H}^*(\Sigma)$.

- The homeomorphism will be represented as a unitary operator

$$U_f : \mathcal{H}(\Sigma) \to \mathcal{H}(\Sigma)$$

- Finally, partition function on manifold **M** is

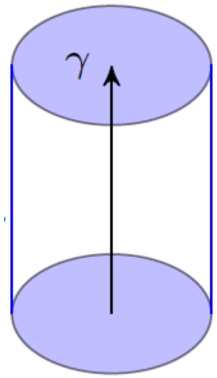
$$Z(M) = \langle\Psi_{M_2}|U_f|\Psi_{M_1}\rangle$$

- Where

$$Z(M) = \int [\mathcal{D}A] \mathrm{e}^{\mathrm{i}S}$$

Wilson loop expectation value on S^3

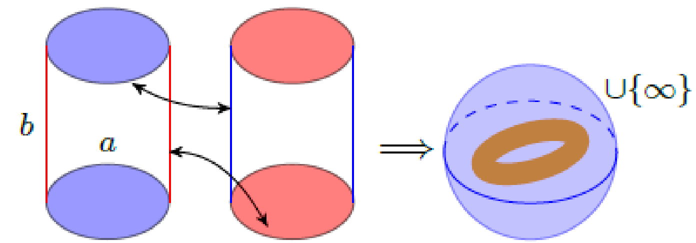
- We consider the group is SU(2)
- We know that a S^3 manifold can be consider as a gluing of 2 solid tori
- The common boundary is $\Sigma = T^2$
- We denote the state $\psi_{\gamma,k}$ in the Hilbert space can be described by placing a Wilson line inside the torus along b-circle transform in the representation of weight $\gamma / 2$.
- With $\gamma = 0$ (trivial representation) we do not insert any Wilson loop.



- Using canonical quantization of Chern-Simons theory on the torus $T^2 \times R$, in the case of group SU(2), we can derive metric elements (The calculation can do in geometrical quantization and also in conformal field theory)

$$S_{ab} = \langle \psi_{a,k} | S | \psi_{b,k} \rangle = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi(a+1)(b+1)}{k+2} \right)$$

- In which S is the S-transformation that we mentioned before.
- After gluing 2 tori using S-transformation, we have S^3 manifold with a Hopf link including 2 Wilson loop with in different representations.



- Further more, we also can derive that the transformation T just gives state a phase

$$T\left|\psi_{\gamma,k}\right\rangle = e^{\frac{2\pi i}{k+2}C_2(\gamma)-\frac{2\pi i c}{24}}\left|\psi_{\gamma,k}\right\rangle$$

- In which C_2 is quadratic Casimir of the representation and c is the central charge of WZW model, in SU(2) we have

$$c = \frac{k}{k+2}$$

- From that we have some expectation value of Wilson loop in S^3
- The partition function as gluing 2 solid tori without any Wilson loop.

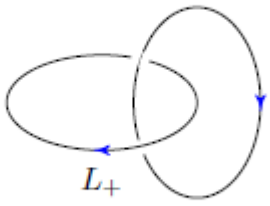
$$Z_{S^3} = \langle \psi_{0,k} | S | \psi_{0,k} \rangle = S_{00} = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right).$$

- Expectation value of a Wilson loop in representation with spin 1/2

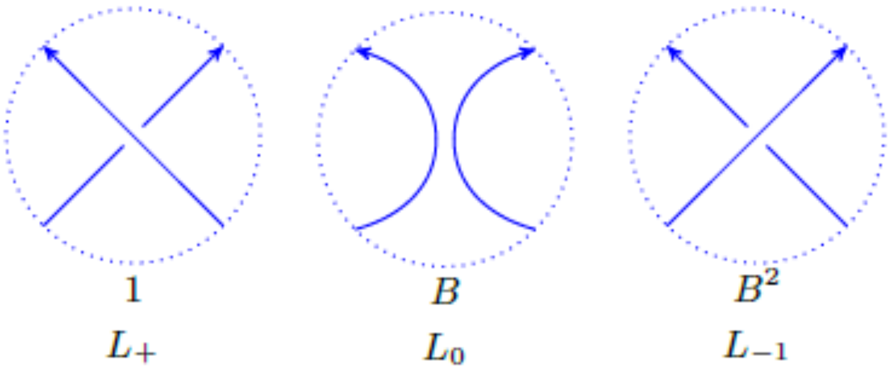
$$\langle \bigcirc \rangle = \frac{Z_{L_{1,0}}}{Z_{S^3}} = \frac{S_{10}}{S_{00}} = q^{1/2} + q^{-1/2}, \quad q = \exp \frac{2i\pi}{k+2}$$

- Expectation value of a Hopf link with 2 spin ½ Wilson loop

$$\langle \text{Hopf} \rangle = \frac{\langle \psi_{1,k} | S | \psi_{1,k} \rangle}{S_{00}} = \frac{S_{11}}{S_{00}} = (q^{1/2} + q^{-1/2})(q + q^{-1}).$$



- As the link gets more complicated, the direct computation becomes harder. Instead, we have the skein relation of a link in representation spin=1/2



- Using surgery operators we also can derive the relation

$$q^{3/2}L_- - q^{3/4}L_0(q^{1/4} - q^{-3/4}) - L_+q^{-1/2} = 0$$

- **Jones polynomial with q=t !**

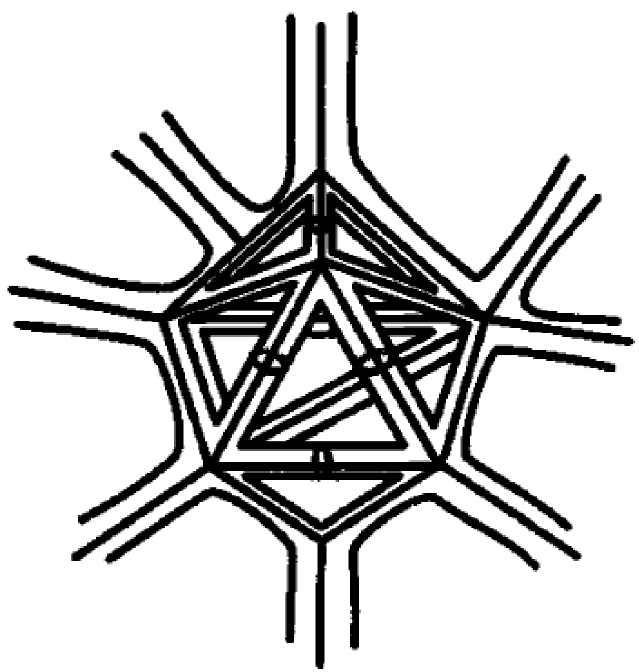
Conclusion

- We had some fun with links and knots
- Surgery is not always hurt
- Jones polynomials make the calculations easy

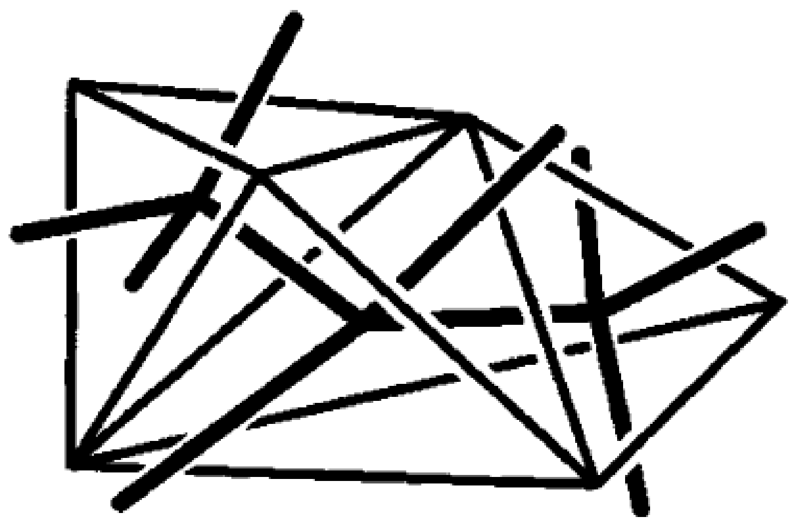
Appendix

- Heegard splitting

- **Heegard splitting**
- ***The idea behind:***
- Every compact orientable 3-manifold has a finite tetrahedralization (splitting into tetrahedrons).
- The one-skeleton is the union of all vertices and edges of tetrahedralization, if we H_1 thicken it   up, we has a handle body within 3-manifold



- We also have the **dual graph** H_2 of the tetrahedralization, it is also handle body with the same genus as the tetrahedralization, if we thicken both of them enough, they will fill up the 3-manifold **M** and share the same boundary as gluing map.



- The gluing that we've seen is Heegard splitting of manifold M into H_1 and H_2.