JOURNAL CLUB - ANOMALY INFLOW

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(1)

→ Review of last time (descent)
→ Basic model - axion electrodynamics
  Application: IQHE?
→ ~~Application: D5-branes~~?

Review

Descent equations:

$$\begin{tikzcd}
 & & \Omega^0_{2n+1} \arrow[r,"d"] \arrow[d,"\delta"] & \Omega^0_{2n+2} = \mathrm{tr}(F^{n+1}) \arrow[r,"d"] \arrow[d,"\delta"] & 0 \\
 & \Omega^1_{2n} \arrow[r,"d"] \arrow[d,"\delta"] & \delta\Omega^0_{2n+1} \arrow[r,"d"] \arrow[d,"\delta"] & 0 \\
\Omega^2_{2n-1} \arrow[r] \arrow[d,"\delta"] & \delta\Omega^1_{2n} \arrow[r,"d"] \arrow[d,"\delta"] & 0 \\
\delta\Omega^2_{2n-1} \arrow[r,"d"] & 0
\end{tikzcd}$$

...

Consistent gauge anomaly: $\delta\Gamma = \mathcal{A} = \int d^dx\, \Omega^1_{2n}$,

Example: $\Omega^0_4 = \frac{1}{8\pi^2} F^2$

(abelian gauge th.) $\Rightarrow \Omega^0_3 = \frac{1}{8\pi^2} A F$

$\Rightarrow \Omega^1_2 = \frac{1}{8\pi^2} \Lambda F$ (up to addition of variation of local counterterms).

Basic model - axion electrodynamics

Take the following theory in 3+1 d:

$$\mathcal{L} = -\frac{1}{4e^2} F^{\mu\nu}F_{\mu\nu} + i\bar{\Psi}\not{D}\Psi + |\partial_\mu\phi|^2 - \bar{\Psi}(\phi_1 + i\gamma_5\phi_2)\Psi - V(\phi),$$

$$\phi = \phi_1 + i\phi_2, \quad V(\phi) = \lambda(|\phi|^2 - v^2)^2$$

This theory has a non-anomalous gauge symmetry, $\Psi \to e^{i\alpha(x)}\Psi$, $A_\mu \to A_\mu + \partial_\mu\alpha$.

However, consider inserting a topological defect in the form of an axion string:

write $\phi = f(\rho)e^{ia}$, with $\rho = \sqrt{x_1^2 + x_2^2}$ (radial cylindrical coordinate), and $f(\rho) \xrightarrow[\rho\to 0]{} 0$, $\xrightarrow[\rho\to\infty]{} v$. $\phi_1 + i\gamma_5\phi_2 = f(\rho)\,e^{ia\gamma_5}$

a can change by $2\pi n$ if we go around the origin once ($\theta \to \theta + 2\pi$), and n can't change continuously, so this is a topological defect. In this background, with $A_\mu = 0$, Ψ equation of motion reads

$i\not{\partial}\Psi = f(\rho)e^{ia\gamma_5}\Psi$. This has chiral 0-modes on the string. To see this, separate the coordinates into $x_{int} = \{x_0, x_3\}$ along the string, $x_{ext} = \{x_1, x_2\}$ $\perp$ to it, and define $\gamma_{int} = \gamma^0\gamma^3$, $\gamma_{ext} = i\gamma^1\gamma^2$ so $\gamma_5 = \gamma_{int}\gamma_{ext}$. Then equation becomes

$$i\gamma^a\partial_a\Psi + i\gamma^1\partial_1\Psi + i\gamma^2\partial_2\Psi = f(\rho)(\cos a + i\gamma_5 \sin a)\Psi.$$

For simplicity, take $a(\theta) = \theta$. Then multiply by $P_\pm = \frac{1 \pm \gamma_5}{2}$ s.t. $P_\pm\Psi = \Psi_\pm$, so equation reads

$$i\gamma^a\partial_a\Psi_\mp + i\gamma^1(\cos\theta - i\gamma^{ext}\sin\theta)\partial_\rho\Psi_\mp = f(\rho)e^{\pm i\theta}\Psi_\pm$$

This has solutions ($\Psi = \Psi_+ + \Psi_-$)

$$\Psi_\pm = \eta_\pm(x_{int}) \exp\left(-\int_0^r f(\sigma)\,d\sigma\right)$$

with $i\gamma^a \partial_a \eta_\pm = 0$

$\gamma^{int}\eta_\pm = -\eta_\pm$ → neg. chirality on string dim.!

$\gamma^{ext}\eta_\pm = \mp\eta_\pm$

$\eta_\pm = -i\gamma^1\eta_\mp$

For Weyl rep. of γ matrices, this is $\eta_+ = \begin{pmatrix}0\\0\\0\\\chi\end{pmatrix}$ $\eta_- = \begin{pmatrix}i\chi\\0\\0\\0\end{pmatrix}$ with $(\partial_0 - \partial_3)\chi = 0$,

so these modes travel up the string at the speed of light. Note the solution dies exponentially away from the string, so it can be said to live on the string. For a general n, there will be n of these.

Why is this significant?

$\gamma^{int}\eta_\pm = -\eta_\pm$ ⇒ these zero-modes can be interpreted as excitations of an effective theory on the string, which is an anomalous because they are chiral, so we get

$$\partial_a j^a = -\frac{e}{8\pi}\epsilon^{ab}F_{ab}.$$

This means that if we turn on an electric field along the z-direction, $F^{03} = \epsilon$, $\partial_a j^a = \frac{e}{4\pi}\epsilon$, ⇒ charge is appearing on the string! But the theory was not anomalous. Where is it coming from?

The remaining massive modes of the fermion must resolve this, mediating interaction between axion field and background gauge/gravitational fields.

Start by performing a chiral rotation on the fermions, (4)

$$\Psi \to e^{ia\gamma_5/2}\Psi$$

$$\bar\Psi \to \bar\Psi e^{ia\gamma_5/2},$$

which transform the fermion-scalar coupling

$\bar\Psi f e^{ia\gamma_5}\Psi \to \bar\Psi f \Psi$ (far from the string, $\bar\Psi\mu\Psi$, just a mass term)
and introduce a interaction $-\frac{1}{2}\bar\Psi\gamma^5 \partial\!\!\!/ a\Psi$ which doesn't give any current. However, this chiral transformation has a non-trivial Jacobian, which we learned to calculate in Chris's talk:

$$S_{Jac} = -\frac{1}{8\pi^2}\int a\, F\wedge F = -\int a\,\Omega^0_4.$$

a is only defined mod 2π, so it's better to write this as

$$S_{Jac} = \int da\ \Omega^0_3$$

Under gauge transformations,

$$\delta S_{Jac} = \int da\ \delta\Omega^0_3 = \int da\ d\,\Omega^1_2 = -\int d^2a\ \Omega^1_2.$$

Recall that a is not well-defined, so $d^2a \neq 0$. In fact,

$$d^2a = 2\pi n\,\delta(x)\delta(y),\ \text{so}$$

$$\delta S_{Jac} = -\int_{string} 2\pi n\ \Omega^1_2.$$

Similarly, from the descent equations, we can find

$$\delta S_{0\text{-modes}} = \int_{string} 2\pi n\ \Omega^1_2$$

So the full theory is non-anomalous as described. S_{Jac} also contributes a piece to the current,

$J^\mu \sim \varepsilon^{\mu\nu\rho\sigma}\partial_\nu\theta F_{\rho\sigma}$, which gives the right amount of electric charge flowing onto the string so the total theory obeys charge conservation.

A subtlety:

- the current written above actually overcompensates: in 2d $U(1)$, there is a factor of 2 between the two $\partial_a j^a$;
- this is the difference between the consistent and covariant anomaly: the current coming from the zero-modes is in consistent form, since it comes from varying the action for zero modes, while the current from the bulk action is covariant (it doesn't come from a 2d eff. action);
- another way to see this is that the action itself is only well-defined where $f \neq 0$, so not on the string, and our manipulations are not justified. Taking into account the profile of f, we obtain an extra term that interpolates between the consistent and covariant forms. See NACULICH; HARVEY & RUCHAYSKIY

There is a <u>general picture</u>:

take a general theory in D ~~d~~$=2n+1$ or $2n+2$ dimensions, with a submanifold of dim. $2n$ M^{2n}, where chiral zero-modes live.

These chiral zero modes have a gauge anomaly

$$\delta S_{0\text{-modes}} = \int_{M^{2n}} I^1_{2n} \propto \int \Omega^1_{2n}.$$

Then for $D=2n+1$, this means the action in the bulk should have a term

$$S_{bulk} \sim \int_{M^D} \Omega^0_{2n+1} \quad \left(\text{or for } D=2n+2, \quad S_{bulk} = \int_{M^D} da\, \Omega^0_{2n+1}\right),$$

so $\delta S_{bulk} \sim \int \Omega^1_{2n}$ from the descent equations, and in the full theory

$$\delta S_{bulk} + \delta S_{0\text{-modes}} = 0$$

Quantum Hall Effect

Take a 2+1d space with 1+1d boundaries. The effective field theory for gauge fields on the bulk can be written

$$S_{bulk} = \frac{\sigma}{2}\int d^3x \left[\varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + j^\mu A_\mu\right],$$ which gives

$$j^\mu = -\frac{\sigma}{2}\varepsilon^{\mu\nu\rho} F_{\nu\rho}.$$ (For $\mu = 1, 2$, this gives $j^1 = \sigma_{xy} E_2$, $j^2 = -\sigma_{xy} E_1$)

This has a gauge variation due to the boundary,

$$\delta S_{bulk} = \sum_{edges} -\frac{\sigma_{xy}}{2}\int_{bound} \varepsilon\, dA$$

However, there are also chiral massless modes localized to the boundary. For n_α modes, these have an anomaly

$$\delta S_{\alpha\text{-modes}} = \pm n_\alpha \frac{e^2}{2\pi}\int \varepsilon\, dA$$

$\hookrightarrow$ modes on α^{th}-edge.

For the full theory to be gauge-invariant, we must have $n \equiv \pm n_1 = \pm n_2$ and $\sigma_{xy} = n\frac{e^2}{2\pi}$, the Hall quantization condition!