Euler equation and Navier-Stokes equation

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ABSTRACT: This is the note prepared for the Kadanoff center journal club. We review the basics of fluid mechanics, Euler equation, and the Navier-Stokes equation. The stability of the solution is discussed by adapting Landau's original argument.

Contents

1 Introduction

In this quarter, we wish to cover the some basics of turbulence in the first 3 talks. As a warm up, the first talk aims to cover some building blocks in the field of fluid mechanics. Fluid dynamics describes collective motion of enormous particles macroscopically. Therefore, we no longer use the coordinates of each individual. Instead, the velocity field $\mathbf{v}$ as a function of spacetime is the desired variable. On top of the velocity, we need 2 thermodynamic variables and the equation of state to complete the problem. In many cases, we choose the fluid density ρ and pressure p.

In d space dimensions[1], we have $(\mathbf{v}, \rho, p)$ up to $d + 2$ variables. The equation of state provides 1 constraint. To fully solve for them, we need extra $d + 1$ equations. We will see shortly they are provided by conservation laws and force balancing equations. In the following sections we are going to introduce them for different types of fluids mainly following the book by Landau [1]. Also useful is the classic textbook by Batchelor [2].

The talk and the note is organized as follows. We start will the ideal fluid, in which case dissipation is absent, to set up the general idea of fluid dynamics. Following the idea, we introduce the viscous term to the stress tensor. By arguing the general form it can has, we derive the Navier-Stokes equation.

[1] In this note, we consider the case $d = 2$ or 3.

Basic properties, energy dissipation and law of similarity, are discussed. Following this section, we pave the path toward turbulence by studying the stability of a viscous, incompressible and steady flow, giving an estimate of the critical Reynolds number. Then the we will ended with some open problems.

2 Ideal Fluid

2.1 Continuity Equation

The first equation is given by the conservation of particle number. The idea is essentially the same as what we studied in electromagnetism and we will merely give an one-minute review. Looking at a given region V, the rate of change for the particle number in V is

$$\frac{\partial}{\partial t}\int_V d^d x\, \rho. \tag{2.1}$$

Accordingly, the particles flowing out through ∂V is the surface integral

$$\int_{\partial V} d\mathbf{a}\cdot\rho\mathbf{v} = \int_V d^d x\, \nabla\cdot(\rho\mathbf{v}). \tag{2.2}$$

Equating (2.1) to $-$(2.2), we have the continuity equation

$$\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\mathbf{v}) = 0. \tag{2.3}$$

2.2 Momentum Equation

The second hydrodynamic equation is Newton's second law. In a fluid, in addition to the total applied force, there is internal momentum transfer due to pressure p, which is

$$-\int d\mathbf{a}\, p = -\int d^d x\, \nabla p. \tag{2.4}$$

Therefore $-\nabla p$ is recognized as the force density acting on fluid. According to Newton's law, it should be the source of the rate of change of the momentum

$$\rho\frac{d\mathbf{v}}{dt} = \rho\Big(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\Big)\mathbf{v}. \tag{2.5}$$

This time derivative accounts for the fact that the fluid particles also move in space. It can actually be regarded as a kind of derivative, using which one measures the force along the velocity streamline. The second term $\mathbf{v}\cdot\nabla\mathbf{v}$, the inertial force, is the connection term owing to moving with the flow. Identifying the rate of velocity change and pressure, we have

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}(\cdots), \tag{2.6}$$

where $(\cdots)$ refers to other plausible external forces acting on the fluid with no dissipation.
We note that the continuity equation and Newton's second law give $d+1$ equations. Since there is no

heat production owing to dissipation. The fluid motion can be considered adiabatic. The equation of state then is given by isentropic condition

$$\frac{ds}{dt} = \frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s = 0. \tag{2.7}$$

An alternative form of conservation can be written in terms of the entropy density ρs.

$$\begin{aligned} &\frac{\partial}{\partial t}(\rho s) = s\frac{\partial \rho}{\partial t} + \rho\frac{\partial s}{\partial t} = -(s\nabla \cdot (\rho\mathbf{v}) + \rho\mathbf{v} \cdot \nabla s) = -\nabla \cdot (\rho s\mathbf{v}) \\ \Rightarrow &\frac{\partial}{\partial t}(\rho s) + \nabla \cdot (\rho s\mathbf{v}) = 0. \end{aligned} \tag{2.8}$$

In this expression, $\rho s\mathbf{v}$ is interpreted as the entropy flux density.

2.3 Momentum Equations as Conservation Laws

In the previous paragraphs we have derived a set of equations that determine $(\mathbf{v}, \rho, p)$ completely. In this paragraph, we are about to spend some lines on writing the Newton's second law as a conservation law in the spirit of $\partial_t(\cdots)+\nabla\cdot(\cdots) = 0$ with the help of equation of continuity. Now the momentum density ρv_i and its temporal partial derivative are given by

$$\begin{aligned} &\frac{\partial(\rho v_i)}{\partial t} = \rho\frac{\partial v_i}{\partial t} + v_i\frac{\partial \rho}{\partial t} \\ =& -\rho(v_j\partial_j)v_i - \partial_i p - v_i\partial_j(\rho v_j) = -\partial_j(\delta_{ij}p + \rho v_i v_j) = -\partial_j\Pi_{ij}. \end{aligned} \tag{2.9}$$

Recall the material in Griffiths chapter 8, Π_{ij} is the momentum flux density. In the following, instead of Euler equation, we will implement this one as we generalize the discussion to viscous fluids.

3 Viscous Fluid

3.1 Equation of Motion

In this section consider the existence of viscosity, or the internal friction between particles constitute the fluid. Including this effective does not change the conservation of particle number and consequently does not change the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0. \tag{3.1}$$

What is changes is the momentum equation and the proper equation of state. Let us recall the momentum equation derived in the last section

$$\text{ideal}: \ \frac{\partial}{\partial t}\rho v_i = -\partial_j\Pi_{ij}, \ \Pi_{ij} = p\delta_{ij} + \rho v_i v_j. \tag{3.2}$$

It represents the reversible momentum transfer owing to the action of pressure and mass. In the presence of viscosity, an additional term is needed to represent the irreversible momentum transfer from large $|\mathbf{v}|$ to small $|\mathbf{v}|$. We write

$$\text{viscous}: \ \Pi_{ij} \to p\delta_{ij} - \sigma'_{ij} + \rho v_i v_j := -\sigma_{ij} + \rho v_i v_j. \tag{3.3}$$

The stress tensor σ is responsible for the momentum transfer not directly due to mass, incorporating conservative part and dissipative part.
A immediate question is that what is the general form of σ' in terms v_i or its derivative. We want to argue the general form from 2 physical observations.
The first one is that the internal friction does not manifest in uniform flow. It must be proportional to $\partial_i v_j$, to the first order in derivative expansion. Given such a tensor, we can decompose it into the trace part, symmetric and traceless part, and the skew symmetric part.

$$\partial_i v_j \to \delta_{ij}\nabla\cdot\mathbf{v},\ \partial_i v_j + \partial_j v_i - \frac{2}{d}\nabla\cdot\mathbf{v},\ \text{and } \partial_i v_j - \partial_j v_i. \tag{3.4}$$

The second observation is that this effect also disappears under rigid rotation $\mathbf{v} = \mathbf{\Omega}\times\mathbf{r}$. It implies only symmetric terms are retained. Consequently, we have the general decomposition

$$\sigma'_{ij} = \eta\Big(\partial_i v_j + \partial_j v_i - \frac{2}{d}\delta_{ij}\nabla\cdot\mathbf{v}\Big) + \zeta\delta_{ij}\nabla\cdot\mathbf{v}, \tag{3.5}$$

where η and ζ are coefficients of viscosity. It can be shown that both of them are positive. Using the equation of motion $\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0$, the above equation can be written as

$$\rho\Big(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\Big)v_i = -\partial_i p + \partial_k\sigma'_{ik}. \tag{3.6}$$

There is no reason a priori to forbidden η and ζ from being fields. Nonetheless, in the following, we will be treating them as constants. Therefore, the momentum equation becomes

$$\rho\Big(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\Big)\mathbf{v} = -\nabla p + \eta\nabla^2\mathbf{v} + \Big(\zeta + \frac{1}{d}\eta\Big)\nabla(\nabla\cdot\mathbf{v}). \tag{3.7}$$

Equations (3.6) and (3.7) are the Navier-Stokes equation.

3.2 Incompressible Fluid

We have modified the momentum equations in the presence of viscosity. Due to dissipation and the heat produced. There is no reason to assume adiabatic process $ds/dt = 0$. A model dependent equation of state has to be proposed to provide with sufficient constraints. For the case we will investigate in the rest of the talk, we narrow our attention to a certain kind of fluid, the incompressible flow defined by $\rho = \text{const}$. As such, ρ is no longer regarded as an unknown. The continuity equation reduces to $\nabla\cdot\mathbf{v} = 0$. The momentum equation becomes

$$\Big(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\Big)\mathbf{v} = -\frac{\nabla p}{\rho} + \nu\nabla^2\mathbf{v} \tag{3.8}$$

with $\nu = \eta/\rho$ being the kinematic viscosity.
To make the equation simpler, we take the curl of the full equation. Note that

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{1}{2}\nabla v^2 - \mathbf{v}\times(\nabla\times\mathbf{v}) \tag{3.9}$$

$$\Rightarrow \nabla\times(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\times[\mathbf{v}\times(\nabla\times\mathbf{v})]. \tag{3.10}$$

Thus, we arrive at

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{v}) = \nabla \times [\mathbf{v} \times (\nabla \times \mathbf{v})] + \nu \nabla^2 (\nabla \times \mathbf{v}). \tag{3.11}$$

On the other hand, taking the divergence of the equation gives us a Poisson type equation

$$-\nabla^2 p = -\rho \nabla \cdot (\mathbf{v} \cdot \nabla)\mathbf{v} = \rho \partial_i \partial_k (v_k v_i). \tag{3.12}$$

Using (3.11), the velocity profile can be solved. Plugging the profile into (3.12), we can solve for the variable p.
To completely determine the solution, boundary conditions are required. Contrast to the ideal fluid, here we demand $\mathbf{v} = 0$ over the boundary owing to internal friction.

3.3 Energy Dissipation

Next we discuss the rate at which the energy is dissipated. The kinetic energy of the fluid in a region V is given by

$$E = \frac{1}{2} \int_V d^d x \, \rho v^2. \tag{3.13}$$

For incompressible fluid, the rate of change $\dot{E}$ is proportional to $\frac{\partial}{\partial t}(v^2)$. Dotting (3.6) by $\mathbf{v}$,

$$\rho \frac{1}{2} \frac{\partial}{\partial t} v^2 = -\rho \mathbf{v} \cdot (\mathbf{v} \cdot \nabla)\mathbf{v} - \mathbf{v} \cdot \nabla p + v_i \partial_j \sigma'_{ij}. \tag{3.14}$$

$v_i \partial_j \sigma'_{ij} = \partial_j (v_i \sigma'_{ij}) - \sigma'_{ij} \partial_j v_i$. Using $\nabla \cdot \mathbf{v} = 0$, $v_i v_i \partial_i v_j = \frac{1}{2} v_i \partial_i v^2 = \frac{1}{2} \partial_i (v_i v^2)$ and $v_i \partial_i p = \partial_i (v_i p)$. Thus,

$$\dot{E} = \frac{\rho}{2} \int d^d x \, \frac{\partial}{\partial t} v^2 = -\int_{\partial V} d\mathbf{a} \cdot \left[\rho \mathbf{v} \Big(\frac{1}{2} + \frac{p}{\rho} \Big) - \mathbf{v} \cdot \boldsymbol{\sigma}' \right] - \int_V d^d x \, \sigma'_{ik} \partial_k v_i. \tag{3.15}$$

As we push ∂V to infinity and write $\sigma'_{ik} = \eta(\partial_i v_k + \partial_k v_i)$,

$$\dot{E} = -\frac{\eta}{2} \int_V d^d x \, (\partial_i v_j + \partial_j v_i)^2. \tag{3.16}$$

Since $\dot{E} < 0$ and the integrand is positive definite, physically we see $\eta > 0$.

3.4 Law of Similarity

Let us think about what we can say using the scaling property of the Navier-Stokes equation. In this section we give the argument for steady and incompressible flows and look at the problems in which the fluids flow past solid bodies of the same shape with different length scales.
Under this circumstance, the only parameter appearing in the equation of motion is the kinematic viscosity ν. The main stream has a characteristic velocity u and the boundary conditions give a characteristic linear dimension ℓ. Their dimensions are

$$[\nu] = \frac{L^2}{T}, \; [u] = \frac{L}{T}, \; [\ell] = L. \tag{3.17}$$

From these informations we can define a dimensionless parameter, the Reynolds number, as

$$\mathrm{R} = \frac{u\ell}{\nu}. \tag{3.18}$$

Other dimensionless quantities are essentially functions of R. R can be given a direct physical interpretation as we write

$$\mathrm{R} = \frac{u\ell}{\nu} = \frac{u^2/\ell}{\nu u/\ell^2} \sim \Big|\frac{\mathbf{v}\cdot\nabla\mathbf{v}}{\nu\nabla^2\mathbf{v}}\Big|, \tag{3.19}$$

which can interpreted as the competition between the inertial force, the non-linear term, and the viscous force.
It turns out that we can measure $\mathbf{v}$ and coordinates $\mathbf{x}$ in units of u and ℓ respectively. Formally

$$\frac{\mathbf{v}}{u} = \mathbf{v}(\mathbf{x}/\ell, \mathrm{R}). \tag{3.20}$$

The conclusion is if 2 incompressible and steady flows with the same Reynolds number flowing past similar solid bodies, they flows are then given by the same function of $\mathbf{x}/\ell$. The same implication applies to the ratio $p/(\rho u^2)$ and other physical quantities. This is the law of similarity.
Generalizing this law as more dimensionful physical quantities come into play in possible. For example, as gravity is significant in the problem of interest, another dimensionless parameter, the Froude number,

$$\mathrm{F} = \frac{u^2}{\ell g} \tag{3.21}$$

would be defined beside the Reynolds number. Flows are similar as they have the same Reynolds number and Froude number.
It can even be generalized to non-steady flows, in which a time scale is in the game. For instance, as we immerse an oscillating solid body into the fluid, its inverse oscillation frequency τ introduces a time scale. As such, the so-called Strouhal number is defined as

$$\mathrm{S} = \frac{u\tau}{\ell}. \tag{3.22}$$

Flows with the same S and R exhibit similar motion.
To end this section we note that the story is quite different for ideal fluids, whose $\eta = 0$. In those cases, the Euler equation is essentially scale invariant.

4 Stability Condition

Either Euler equation or Navier-Stokes equation is nonlinear. In particular, there is a parameter ν, or equivalently, R, in the Navier-Stokes equation, which we can tune and drive the equation from linear regime to nonlinear one. In the extremely viscous case $\nu \to \infty$, or $\mathrm{R} \to 0$, the equation is dominated by dissipation and linear. Perturbative study can be carried out. As we look at a range of R, we may

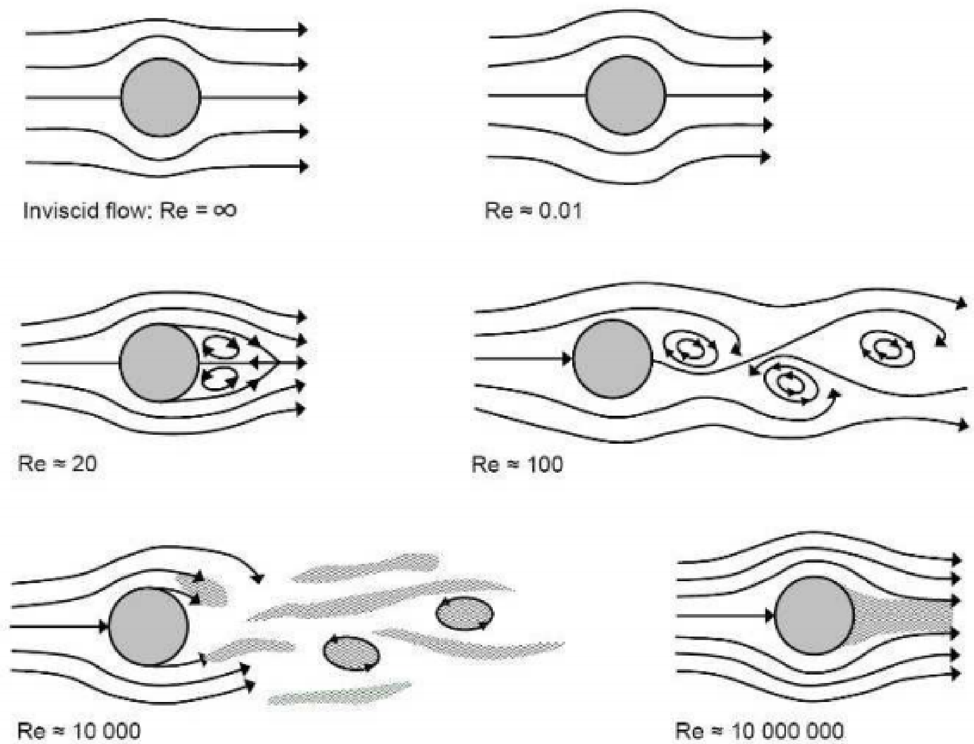

Figure 1. This figure depicts the stream lines of fluids of different R flowing past a cylinder. Going from small R to large ones, we see the stream lines go from Laminar to turbulent. This figure is adopted from Ref. [5]. To be more elaborated, as R ~ 10, vortices start forming. As R $\to 100$, those vortices form vortex streets. Turbulent phenomenon occurs as R reaches 10^4.

first look at their phenomenologies illustrated in figure 1.

From figure 1, we understand as R increases beyond a threshold, the fluid becomes turbulent, which roughly speaking is the chaotic behavior of fluid. Here we try to establish this picture by examining the fate of a little deviation from a given steady solution $\mathbf{v}_0(\mathbf{x})$. In our experience, owing to nonlinearity, a little deviation added to a solution may lead to dramatic change in its nature, and leads to the investigation of stability.
We write

$$\mathbf{v}(t,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) + \delta\mathbf{v}(t,\mathbf{x}) \tag{4.1}$$

$$p = p_0 + \delta p. \tag{4.2}$$

To first order in δ, we have

$$\frac{\partial}{\partial t}\delta\mathbf{v} + (\mathbf{v}_0\cdot\nabla)\delta\mathbf{v} + (\delta\mathbf{v}\cdot\nabla)\mathbf{v}_0 = -\rho^{-1}\nabla\delta p + \nu\nabla^2\delta\mathbf{v},\ \nabla\cdot\delta\mathbf{v} = 0. \tag{4.3}$$

At this order $\delta\mathbf{v}$ satisfies a linear equation. It turns out we can look at the normal modes $\delta\mathbf{v} \sim e^{-i\omega t}$ and the flow is said to be stable if $\mathrm{Im}[\omega] < 0$.
More precisely, we say as $\mathrm{R} < \mathrm{R_c}$, the critical Reynolds number, all ω's have negative imaginary part. At the critical point, one ω becomes real, and as it slightly goes larger than $\mathrm{R_c}$, its imaginary part becomes positive. We write this frequency as

$$\omega = \omega' + i\omega'',\ \omega'' \ll \omega' \tag{4.4}$$

and the deviation assumes the form

$$\delta\mathbf{v}_\omega = A(t)\mathbf{f}(\mathbf{x}),\ A(t) \propto e^{\omega'' t} e^{-i\omega' t}. \tag{4.5}$$

We comment that this expression holds true at small time. If we are to consider the evolution of the magnitude averaged over a period $(\omega'')^{-1} \gg \tau \gg (\omega')^{-1}$, to the leading order, we would naively write $\frac{d}{dt}|A|^2 = 2\omega''|A|^2$. Landau conjectured a subsequent term

$$\frac{d}{dt}|A|^2 = 2\omega''|A|^2 - \alpha|A|^4, \tag{4.6}$$

where α is the Landau constant which can be positive or negative. Let us argue for the case that $\omega'' > 0$ and $\alpha > 0$, $|A|^2$ assumes a finite limit value and therefore the solution is not stable. Solving the above equation, we obtain

$$\frac{1}{|A|^2} = \frac{\alpha}{2\omega''} + \mathrm{const} \times e^{-2\omega'' t} \tag{4.7}$$

and $|A|^2$ has an asymptotic value $2\omega''/\alpha$. It may concerns one that as we plug the asymptotic value back into the differential equation, each term on the right-hand of (4.6) is of the same order. Consequently, it seems not reasonable to drop $\beta|A|^6$ or any other higher order terms. A more proper way we should imagine this process is that equation is valid at small time. As $|A|^2$ increases such that the existing 2 terms are of the same order, the next order should be included self-consistently. Owing to the assumption that ω'' is slightly greater than 0 as R exceeds $\mathrm{R_c}$, if we are slightly above the critical value, we see that

$$\lim |A| \propto \sqrt{(\mathrm{R} - \mathrm{R_c})}. \tag{4.8}$$

It is a reminiscent of Landau's theory of phase transition. To end this section, we provide an estimate for the critical value $\mathrm{R_c}$. Suppose we do not assume the smallness of $\delta\mathbf{v}$. The fully expanded equation reads

$$\frac{\partial}{\partial t}\delta\mathbf{v} + (\mathbf{v}_0 \cdot \nabla)\delta\mathbf{v} + (\delta\mathbf{v} \cdot \nabla)\mathbf{v}_0 + (\delta\mathbf{v} \cdot \nabla)\delta\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{1}{\mathrm{R}}\nabla^2\delta\mathbf{v}. \tag{4.9}$$

Let us dot this equation by $\delta\mathbf{v}$.

$$\delta v^i \frac{\partial}{\partial t}\delta v^i + \delta v^i v_0^j \partial_j \delta v^i + \delta v^i \delta v^j \partial_j v_0^i + \delta v^i \delta v^j \partial_j \delta v^i = -\rho^{-1}\delta v^i \partial_i p + \mathrm{R}^{-1}\delta v^i \partial_j \partial_j \delta v^i. \tag{4.10}$$

The first term becomes $\frac{1}{2}\frac{\partial}{\partial t}\delta v^2$. Moving other terms to the right-hand side, and using, when assuming $\partial_i \delta v^i = \partial_i v_0^i = 0$,

$$\delta v^i v_0^j \partial_j \delta v^i = \frac{1}{2}\partial_j (v_0^j \delta v^2) \tag{4.11}$$

$$\delta v^i \delta v^j \partial_j \delta v^j = \frac{1}{2}\partial_j (\delta v^j \delta v^2), \tag{4.12}$$

we have

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\delta v^2 = -\delta v^i \delta v^j \partial_j v_0^i - \mathrm{R}^{-1}\partial_j \delta v^i \partial_j \delta v^i \\ + \partial_j \Big(\mathrm{R}^{-1}\delta v^i \partial_j \delta v^i - \rho^{-1}\delta v^j p - \frac{1}{2}\delta v^2 (v_0^j + \delta v^j)\Big).\end{aligned} \tag{4.13}$$

Integrating the whole equation over the domain, the total derivative vanishes owing to boundary conditions. The left-hand side is the rate of change of the kinetic energy of perturbation.

$$\frac{\partial}{\partial t}\delta E = -\int d^d x \, \delta v^i \delta v^j \partial_j v_0^i - \frac{1}{\mathrm{R}}\int d^d x \, \mathrm{tr}[\nabla \mathbf{v}(\nabla \mathbf{v})^T] := T - \frac{D}{\mathrm{R}}. \tag{4.14}$$

It provides with us a lower bound for R. If

$$\mathrm{R} < \min \frac{D}{T} \leq \frac{D}{T}, \tag{4.15}$$

we see $\partial_t \delta E < 0$ strictly and thus it flow is stable.

5 Open Questions

In the previous section, we have seen the stability of the solution fails as the Reynolds number exceeds a critical value. In particle, the limit of $\mathrm{R} \to \infty$ is identified as *fully developed turbulence.* It corresponds to, formally, $\nu \to 0$ at fixed $u\ell$ and it will be the theme in the next talk.

Over the previous sections, we discuss the physical essence of the Euler and Navier-Stokes equations. In this section we would like to list some mathematical problems that remain open. To motivate some of them, we note that the fully developed turbulence features irregular behaviors of the solutions to Navier-Stokes equation. The following are some long standing problems, existence and smoothness of Navier-Stokes solutions on $\mathbb{R}^3$ or $\mathbb{T}^3$, and the breakdown of these solutions on $\mathbb{R}^3$ or $\mathbb{T}^3$, given a specific class of initial data and constraints such as smoothness and energy bound [4]. Partial progress has been made by Leray for weak solutions. Quantitatively, a function $\mathbf{v}$ is a weak solution to the incompressible Navier-Stokes equation if

$$\int d^d x \left(\mathbf{v}\cdot\frac{\partial}{\partial t} + \mathbf{v}\cdot(\mathbf{v}\cdot\nabla) + \nu \mathbf{v}\cdot\nabla^2 + p\nabla\cdot\right)\boldsymbol{\phi} = 0, \tag{5.1}$$

$$\int d^d x \, \mathbf{v}\cdot\nabla\psi = 0, \tag{5.2}$$

where $\boldsymbol{\phi}$ (subject to $\nabla\cdot\boldsymbol{\phi} = 0$) and ψ are compactly supported test functions. Leray has shown in existence of weak solutions $(\mathbf{v}, p)$, whereas the uniqueness is yet to be known.

6 Summary

To summarize, in this talk we have covered the basic hydrodynamic equations, Euler equations and Navier-Stokes equation. In particular, we discuss the stability of solutions as we tune the Reynolds number. We saw from qualitative study that chaotic behavior appears as R exceed a critical value, and analyzed the story using Landau's argument. Finally we point out mathematically the Navier-Stokes equation has a lot of problems to be solved, which is a reason that this equation draws interest across different fields in science.

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