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A twisting and B twisting Models in TQFT

$\mathcal{N}=2$ SCFT is generated by $\{L_n, J_n, G_r^+, G_r^-\}$.

The action of Mirror symmetry:

$J_n \to -J_n$, $L_n \to L_n$, $G_r^+ \to G_r^-$, $G_r^- \to G_r^+$

Define the (left) chiral and antichiral states $|\phi\rangle$:

chiral , if $G^+_{-\frac{1}{2}}|\phi\rangle = 0$

anti-chiral , if $G^-_{-\frac{1}{2}}|\phi\rangle = 0$

Similar for the right sector.

In the $A(X)$ theory, $A(X)$ has BRST charge $Q_A = G^+_{-\frac{1}{2}} + \bar{G}^-_{-\frac{1}{2}}$, which corresponds to (c,a) ring. $Q_A^2 = 0$

In the $B(X)$ theory, the BRST charge $Q_B = G^+_{-\frac{1}{2}} + \bar{G}^+_{-\frac{1}{2}}$, and corresponds to (c,c) ring. $Q_B^2 = 0$

left-moving $\sim$ holomorphic part $\sim T^{1,0}X$

$\mathcal{N}=2$ supersymmetric NLSM $\Phi: \Sigma \to X$,

Σ is Riemannian surface , and X is complex manifold

In terms of chiral superfield, $\Phi^j = \phi^j + \theta^+\psi_+^j + \theta^-\psi_-^j + \dots$

~~The complexified tangent bundle $TX = T^{1,0}X \oplus T^{0,1}X$.~~

positive chiral fermion $\psi_+ \in K^{\frac{1}{2}} \otimes \Phi^*(TX)$

$\psi_+^i \in K^{\frac{1}{2}} \otimes \Phi^*(T^{1,0}X)$ $\qquad$ $\psi_+^{\bar{i}} \in K^{\frac{1}{2}} \otimes \Phi^*(T^{0,1}X)$

negative chiral fermion $\psi_- \in \bar{K}^{\frac{1}{2}} \otimes \Phi^*(TX)$

$\psi_-^i \in \bar{K}^{\frac{1}{2}} \otimes \Phi^*(T^{1,0}X)$ $\qquad$ $\psi_-^{\bar{i}} \in \bar{K}^{\frac{1}{2}} \otimes \Phi^*(T^{0,1}X)$

The action is $S = 2t\int_\Sigma d^2z \left[\frac{1}{2} g_{IJ}(\phi)\, \partial_z\phi^I \partial_{\bar z}\phi^J \right.$
$\left. + i\psi_-^{\bar i} D_z \psi_-^i g_{\bar i i} + i\psi_+^{\bar i} D_{\bar z}\psi_+^i g_{\bar i i} + R_{i\bar i j\bar j}\, \psi_+^i \psi_+^{\bar i} \psi_-^j \psi_-^{\bar j} \right]$

The vector and axial R-rotation acts on superfield as

$U(1)_V : (\theta_+, \theta_-) \to (e^{-i\alpha}\theta_+, e^{-i\alpha}\theta_-) \Rightarrow \psi_\pm \to e^{-i\alpha}\psi_\pm$

$U(1)_A : (\theta_+, \theta_-) \to (e^{-i\beta}\theta_+, e^{i\beta}\theta_-) \Rightarrow \psi_\pm \to e^{\mp i\beta}\psi_\pm$

	Before Twisting			A twist		B twist	
	$U(1)_V$	$U(1)_A$	$U(1)_E$	$U(1)_{E'}$	L	$U(1)_{E'}$	L
ψ_+^i	-1	-1	1	0	$\mathbb{C}$	0	$\mathbb{C}$
$\psi_+^{\bar i}$	1	1	1	2	K	2	K
ψ_-^i	-1	1	-1	-2	$\bar K$	0	$\mathbb{C}$
$\psi_-^{\bar i}$	1	-1	-1	0	$\mathbb{C}$	-2	$\bar K$

$U(1)_V$ is generated by fermion number current $J_V = \bar\psi_+\psi_+ + \bar\psi_-\psi_-$

$U(1)_A$ is generated by dismatch of fermion number $J_A = \bar\psi_+\psi_+ - \bar\psi_-\psi_-$

$$\begin{tikzcd} & (c,c) \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ (c,a) \arrow[dr, leftrightarrow] & & (a,c) \arrow[dl, leftrightarrow] \\ & (a,a) & \end{tikzcd}$$

$\longleftrightarrow$ Mirror sym

— reversal of complex structure of X
(holo $\longleftrightarrow$ anti-holo)

Mirror symmetry exchanges the twist in one sector while leaves the other alone, therefore exchanges A and B models.

In A model, ψ_+^i and $\psi_-^{\bar i}$ are scalars, thereby combines into
$\chi^{i,\bar i} = (\psi_+^i, \psi_-^{\bar i})$, a section of $\Phi^*(TX)$.
$\psi_z^{\bar i} = \psi_+^{\bar i} \sim (1,0)$-form on Σ w/ values in $\Phi^*(T^{0,1}X)$
$\psi_{\bar z}^{i} = \psi_-^{i} \sim (0,1)$-form on Σ w/ values in $\Phi^*(T^{1,0}X)$

The topological transformation law $\delta_A W = -i\alpha\{W, Q_A\}$ is inherited from the supersymmetric transformation law.

$\delta\phi^i = i\alpha\chi^i$, $\delta\phi^{\bar i} = i\alpha\chi^{\bar i}$, $\delta\chi^i = \delta\chi^{\bar i} = 0$

$\delta\psi_z^{\bar i} = -\alpha\partial_z\phi^{\bar i} - i\alpha\chi^{\bar j}\Gamma^{\bar i}_{\bar j\bar m}\psi_z^{\bar m}$, $\delta\psi_{\bar z}^{i} = -\alpha\partial_{\bar z}\phi^i - i\alpha\chi^j\Gamma^i_{jm}\psi_{\bar z}^m$

The action can be written modulo terms that vanish by the ψ EOM:

$$S = it\int_\Sigma d^2z\,\{Q_A, V\} + t\int_\Sigma \Phi^*(\tilde K)$$

$V = g_{\bar i j}(\psi_z^{\bar i}\partial_{\bar z}\phi^j + \partial_z\phi^{\bar i}\psi_{\bar z}^j)$

$\int_\Sigma \Phi^*(\tilde K) = \int_\Sigma d^2z\, g_{i\bar j}(\partial_z\phi^i\partial_{\bar z}\phi^{\bar j} - \partial_{\bar z}\phi^i\partial_z\phi^{\bar j})$ is the pull back of the kahler form on X.

$\int \Phi^*(\tilde K) = 2\pi n$ $(n \in \mathbb{Z}_{\geq 0})$ only depend on cohomology class of $\tilde K$, and homotopy class of the map Φ.

$n = \phi_*[\Sigma] \in H_2(X, \mathbb{Z})$ can be regarded as the winding number of the instanton on X.

The correlation function of Q_A-invariant operators $\mathcal{O}_a$ is

$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle = \int D\phi D\chi D\psi\, e^{-S} \prod_{a=1}^{s} \mathcal{O}_a$ (10)

$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle = \sum_{n\geq 0} \langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_n$ (11), where

$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_n = e^{-2\pi n t}\int_n D\phi D\psi D\chi\, e^{-it\{Q, \int V\}} \prod_{a=1}^{s} \mathcal{O}_a(x_a)$ (12)

is the n-th part which is the path integral of fields of $\deg(n)$.

Except for $e^{-2\pi n t}$, $\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_n$ is independent of t and Q_A invariant.

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What are those operators $\mathcal{O}_a$? To have a covariant zero-form operator, local operators of Q_A-cohomology can only be made up of ϕ and χ.

Given $W = W_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{\bar{j}_1} \wedge \cdots \wedge d\bar{z}^{\bar{j}_q} \in \Omega^{p,q}(X)$

we can associate w/ a local operator

$$\mathcal{O}_W := W_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} \chi^{i_1} \cdots \chi^{i_p} \chi^{\bar{j}_1} \cdots \chi^{\bar{j}_q} . \qquad (13)$$

Since $\{Q_A, \mathcal{O}_W\} = -\mathcal{O}_{dW}$

$$\begin{cases} \text{If } W \text{ is d-closed, then } \{Q_A, \mathcal{O}_W\} = 0. \quad \mathcal{O}_W \text{ is } Q_A\text{-closed.} \\ \text{If } W \text{ is d-exact, then } W = dR \text{ and } \{Q, \mathcal{O}_R\} = -\mathcal{O}_W, \\ \quad \mathcal{O}_W \text{ is } Q_A\text{-exact.} \end{cases}$$

We have the identifications $G^+_{-\frac{1}{2}} \sim \partial$, $\bar{G}^-_{-\frac{1}{2}} \sim \bar{\partial}$

Thus, the 1-to-1 correspondence is set up between Q_A cohomology and d-cohomology.

$$\{\text{physical operators}\} \cong H^*_{DR}(X)$$

$$\mathcal{O}_i \longleftrightarrow W_i \in H^{p_i, q_i}(X). \qquad \begin{cases} q_V = -p_i + q_i \\ q_A = p_i + q_i \end{cases}$$

We try to construct representatives of Q_A-cohomology class of the dual homology cycles.

Let D be a homology cycle of real codimension r. Then its Poincare dual $[D]$ is a cohomology class in $H^r(X)$ which is represented as the delta function r-form supported on $D \in X$. The operator corresponding to $[D]$ is $\mathcal{O}_D$, and is properly normalized. For $x \in \Sigma$, $\mathcal{O}_D(x) = 0$ if $\phi(x) \notin D$.

There is a selection rule in $\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_n$, due to the Q_A-charge anomaly (ghost number anomaly), which is from the fermion zero modes in the measure.

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Index of differential operators that appear in the fermion kinetic terms

$$w_n := \#\chi \text{ zero modes} - \#\psi \text{ zero modes}$$

$$= 2n\, c_1(X) + 2\dim_{\mathbb{C}} X\,(1-g) \qquad g: \text{genus of } \Sigma$$

(These Differential operators are the Dolbeault operators in this case rather than the Dirac operators in the untwisted theory.)

Only if the Q_A charges of $\mathcal{O}_a$ add up to w_n, can $\langle \prod \mathcal{O} \rangle$ be non-zero.

We compute the correlation function using localization principle.

Since the is fermionic symmetry Q_A under which all inserted operators are invariant, the path integral $\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_n$ picks up contributions from the loci where the Q_A-variation vanishes.

Q_A-fixed points: $\delta\phi = 0 \Rightarrow \chi = 0$

$$\delta\psi = 0 \Rightarrow \partial_{\bar{z}} \phi^i = \partial_z \phi^{\bar{i}} = 0 \qquad (20)$$

The map $\phi: \Sigma \to X$ is holomorphic.

Moreover, look at the form of V, the bosonic part of the action is minimized for holomorphic map of Σ to X.

Assume there is no ψ zero modes, and moduli space

$$M_\Sigma(X, n) = \{ \phi: \Sigma \to X \mid \phi \text{ is holomorphic, and } \phi_*(\Sigma) = n \}$$

is a smooth manifold. Then $w_n = \#\chi$ zero modes.

The equation for a χ zero mode is precisely the linearization of the instanton equation (20), so the space of χ zero modes is identified w/ $TM_\Sigma(X, n)$ $\Rightarrow$ $\dim M_\Sigma(X, n) = w_n$ (21)

The integration of non-zero modes gives 1 due to the cancellation of the bosonic and fermionic determinants.

The path integral reduces to the integral over the finite dim space $M_\Sigma(X, n)$, and the measure is $D\chi_0$.

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There is an alternative approach to explaining the anomaly cancellation for Q_A charge. $D\chi_0 = \prod_{k=1}^{w_n} d\chi_{0,k} d\bar{\chi}_{0,k}$, because $\# \chi$ zero modes $= w_n$

when $\mathcal{O}_W$ is expressed as eq. (13). Only when

$\Sigma p_i = \Sigma q_i = k$, (22) can the grassmannian integral not vanish.

Though we have correspondence $W \leftrightarrow \mathcal{O}_W$ ($W \in H^*(X)$),

$\prod_i W_i$ can't be integrated over $\mathcal{M}_\Sigma(X, n)$. So we must pull it back to $\mathcal{M}_\Sigma(X, n)$. Actually $\mathcal{O}_i$ can be identified w/ the pull-back of $\omega_i \in H^*(X)$ by the evaluation map at x_i

$$ev_i : \mathcal{M}_\Sigma(X, n) \to X$$
$$\phi \mapsto \phi(x_i)$$

If ω_i is a r-form on X, $ev_i^*(\omega_i)$ is a r-form.

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_n = e^{-2\pi n t} \int_{\mathcal{M}_\Sigma(X,n)} ev_1^* \omega_1 \wedge \cdots \wedge ev_s^* \omega_s \quad (23)$$

Form degree = dimension of moduli space iff the selection rule (22) is satisfied.

If $[\omega_i]$ are poincare duals of cycles D_i in X, ω_i can be chosen as the delta function supported on D_i. Then the integral (23) can be identified as the number of maps:

$$m_n(D_1, \cdots D_s) = \# \left\{ \phi : \Sigma \to X \;\middle|\; \begin{array}{l} \text{holomorphic}, \ \phi_*[\Sigma] = n \\ \phi(x_i) \in D_i \quad \forall i \end{array} \right\} \quad (24)$$

The total correlation function is

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle = \sum_{n \in H_2(X, \mathbb{Z})} e^{-2\pi n t} m_n(D_1 \cdots D_s) \quad (25)$$

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In the large volumn / large t limit, (25) is dominated by $n = 0$ term. The moduli space $\mathcal{M}_\Sigma(X, 0) \simeq X$, and the evaluation map is the identity map $id_X = ev_i \quad i = 1 \sim s$.

The degree 0 contribution:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle_0 = \int_X \omega_1 \wedge \cdots \wedge \omega_s = \# (D_1 \cap D_2 \cap \cdots \cap D_s)$$

(26) is a quantum deformation of classical intersection numbers.

Now we study B model in the same spirit.

$\eta^{\bar{i}} = \psi_+^{\bar{i}} + \psi_-^{\bar{i}}$ are sections of $\Phi^*(T^{0,1}X)$

$\theta_i = g_{i\bar{i}} (\psi_+^{\bar{i}} - \psi_-^{\bar{i}})$

Combine $\psi_\pm^i$ into an one-form w/ values in $\Phi^*(T^{1,0}X)$

$$\rho^i = \psi_+^i dz + \psi_-^i d\bar{z} \equiv \rho_z^i dz + \rho_{\bar{z}}^i d\bar{z}$$

The topological transformation $\delta_B W = -i\alpha \{Q_B, W\}$:

$$\delta\phi^i = 0, \quad \delta\phi^{\bar{i}} = i\alpha\eta^{\bar{i}}$$
$$\delta\eta^{\bar{i}} = \delta\theta_i = 0, \quad \delta\rho^i = -\alpha\, d\phi^i$$

The action is rewritten as

$$S = it \int \{Q_B, V\} + tW,$$

where $V = g_{i\bar{j}} (\rho_z^i \partial_{\bar{z}} \phi^{\bar{j}} + \rho_{\bar{z}}^i \partial_z \phi^{\bar{j}})$

$$W = \int_\Sigma \left(-\theta_i D\rho^i - \tfrac{i}{2} R_{i\bar{i}j\bar{j}} \rho^i \wedge \rho^j \eta^{\bar{i}} \theta_k g^{k\bar{j}} \right.$$

Local operators $W_{\bar{i}_1 \cdots \bar{i}_p}^{j_1 \cdots j_q} \eta^{\bar{i}_1} \cdots \eta^{\bar{i}_p} \theta_{j_1} \cdots \theta_{j_q}$ correspond to forms $W = d\bar{z}^{i_1} \cdots d\bar{z}^{i_p} W_{\bar{i}_1 \cdots \bar{i}_p}^{j_1 \cdots j_q} \frac{\partial}{\partial z^{j_1}} \cdots \frac{\partial}{\partial z^{j_q}} \in \Omega^{0,p}(X, \Lambda^q T^{1,0}X)$.

One finds $\{Q_B, \mathcal{O}_W\} = -\mathcal{O}_{\bar{\partial}W}$, and $W \to \bar{\partial}W$ gives a natural map $\bigoplus_{p,q=0}^{n} H^{0,p}(X, \Lambda^q T^{1,0}X) \longrightarrow Q_B$ - cohomology.

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Consider correlation function $\langle \mathcal{O}_1 \cdots \mathcal{O}_s \rangle =$
$\int D\phi D\eta D\theta \, e^{-S} \mathcal{O}_1 \cdots \mathcal{O}_s$, where $\mathcal{O}_i \sim \omega_i \in H^{0,p_i}(X, \Lambda^{q_i} T^{1,0} X)$.

Selection rule requires $\sum_{i=1}^{s} p_i = \sum_{i=1}^{s} q_i = n = \dim_{\mathbb{C}} X$ for genus 0.

By localization, Q_B - fixed points obey $\partial_\mu \phi^i = 0$.

It is a constant map, and space of constant maps $\simeq X$.
We may wish to integrate $\omega = \omega_1 \wedge \cdots \wedge \omega_s$ over X. But when the selection rule is satisfied, ω is a $(0,n)$-form in $\Lambda^n TX$.
$\Lambda^n T^{1,0} X$ and $\Omega^d X$ are trivial and isomorphic when X is Calabi-Yau. We can send ω to (n,n)-form via
$\omega \mapsto \langle \omega, \Upsilon \rangle \wedge \Upsilon := \omega^{i_1 \cdots i_n}_{\bar{j}_1 \cdots \bar{j}_n} d\bar{z}^{\bar{j}_1} \cdots d\bar{z}^{\bar{j}_n} \Upsilon_{i_1 \cdots i_n} \wedge \Upsilon$,
when Υ is a holomorphic n-form on X.
Only an (n,n)-form can be integrated naturally. Therefore integration over fermion requires a choice of a section for the holomorphic n-form Υ. It means that topological correlation functions are not really functions, but sections of a suitable bundle on the moduli space of complex structures on Calabi-Yau.

Why should X in B model be Calabi-Yau?
Zero-forms $\eta^{\bar{i}}$, $g^{i\bar{i}} \theta_i$ are sections of $T^{0,1}X$ and the 1-forms ρ^i are sections of $T^{1,0}X$, the fermionic determinant in B model is complex. To make anomaly cancellation, we require $c_1(X) = 0$.