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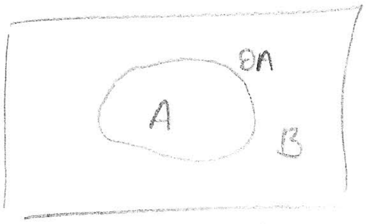
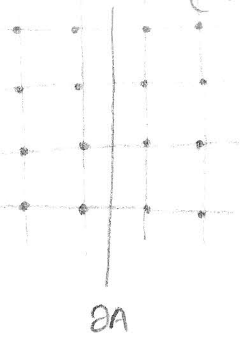
Area Laws for Entanglement Entropy



$S_A = -\mathrm{Tr}\,\rho_A \ln \rho_A$

- Main point: for large class of systems
  (1) EE between regions A+B in ground state
  $\sim \frac{\text{area of boundary}}{\text{cuttoff}^{d-1}}$. d spatial dimension.
- Can be understood heuristically in a few ways
  - since $S_A = S_B$ at least cannot be extensive.
    - must depend only on common feature, such as ∂A
  - if correlations are short length



- little entanglement between for separated spins
- most entanglement concentrated on boundary where have closest spins.

- Mirrors black hole entropy $S_{BH} = \frac{1}{4} M_{pl}^2 A$ A area of horizon.
  (2) - idea: no access to within horizon, so external observer views mixed state that is partial trace $\Rightarrow$ entanglement entropy

(A) How seriously should I be taking this?

- Lets make these ideas more precise

Outline • EE for 2 coupled oscillators.

- Outline of Srednicki's calculation
  - free massless KG field
  - discretize, find finite collection of coupled oscillators
- Path integral formulation
  - n-sheeted riemann surfaces
- Area laws via path integrals
- Exceptions.

2 coupled oscillators

- a la Srednicki, start w/ easy 2-oscillator problem & generalize

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + k_0(x_1^2 + x_2^2) + k_1(x_1 - x_2)^2\right)$$

- seek entanglement between spins 1 & 2 in ground state.
- so let's get the ground state!
  - diagonalize coupling matrix

$$H = \frac{1}{2}\left(p_+^2 + p_-^2 + \omega_+^2 x_+^2 + \omega_-^2 x_-^2\right)$$

where $x_\pm = (x_1 \pm x_2)\frac{1}{\sqrt{2}}$, $\omega_+ = k_0^{\frac{1}{2}}$ $\omega_- = (k_0 + 2k_1)^{\frac{1}{2}}$

  - ground state product of single particle ground states

$$\psi_0(x_1, x_2) \propto \exp\left(\frac{-1}{2}\left(\omega_+ x_+^2 + \omega_- x_-^2\right)\right)$$

$$= \exp\left(-\frac{1}{4}(\omega_+ + \omega_-)(x_1^2 + x_2^2) - \frac{1}{2}(\omega_+ - \omega_-)x_1 x_2\right)$$

- ground state density matrix: $\langle \vec{x} | \rho | \vec{x}' \rangle = \psi_0(x_1, x_2)\psi_0^*(x_1', x_2')$

• take A = spin 1
B = spin 2

$$\langle x_1|\rho_A|x_1'\rangle = \int dx_2\, \Psi_0(x_1,x_2)\Psi_0^*(x_1',x_2) \propto \exp\left[-\gamma(x_1^2+x_1'^2)/2+\beta x_1 x_1'\right]$$

↑ gaussian integral

where $\beta = \frac{1}{4}\frac{(\omega_+-\omega_-)^2}{\omega_++\omega_-}$ $\quad \gamma-\beta = \frac{2\omega_+\omega_-}{\omega_++\omega_-}$

• Now $S_A = -\sum_n p_n \ln p_n$ where p_n are eigenvalues of above operator.

$$\int dx' \rho_A(x,x') f_n(x') = p_n f(x)$$

• ρ_A looks like SHO propagator

so guess $f_n = H_n(\alpha^{\frac{1}{2}}x)\exp(-\frac{1}{2}\alpha x^2)$ $\quad \alpha = (\gamma^2-\beta^2)^{\frac{1}{2}}$

$$\Rightarrow p_n = (1-\xi)\xi^n \quad \text{where} \quad \xi = \frac{\beta'}{1+(1-\beta'^2)^{\frac{1}{2}}} \quad \& \quad \beta' = \frac{\beta}{\gamma}$$

• $S_A = -\sum_n (1-\xi)\xi^n \ln(1-\xi)\xi^n = -\sum_n (1-\xi)\ln(1-\xi)\xi^n - \sum_n (1-\xi)\ln\xi\, n\xi^n$

$= -\ln(1-\xi) - \frac{\xi}{1-\xi}\ln\xi$ where we've used $\sum \xi^n = \frac{1}{1-\xi}$ $\quad \sum n\xi^n = \frac{\xi}{(1-\xi)^2}$

• Can generalize to N oscillators (just gaussian integrals!)

$$H = \frac{1}{2}\sum_{i=1}^N p_i^2 + \frac{1}{2}\sum_{i,j=1}^N x_i K_{ij} x_j \qquad \text{let } K^{\frac{1}{2}} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

if take $A = 1,\dots,n$ $\quad B = n+1,\dots,N$

$$S_A = -\sum_{i=1}^n \left(\ln(1-\xi_i) + \frac{\xi_i}{1-\xi_i}\ln\xi_i\right) \quad \text{where } \xi_i = \frac{\beta_i'}{1+(1-\beta_i'^2)^{\frac{1}{2}}}$$

& β_i' are eigenvalues of

$$\beta' = \frac{1}{2}\left(C-\frac{1}{2}B^TA^{-1}B\right)^{-\frac{1}{2}}(B^TA^{-1}B)\left(C-\frac{1}{2}B^TA^{-1}B\right)^{-\frac{1}{2}}$$