Cascade Phenomenology in Turbulence: Navier-Stokes and MHD

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ABSTRACT: This is the note prepared for the Kadanoff center journal club. In this talk we discuss the spectrum and cascade directions in fully developed hydrodynamic / magnetohydrodynamic turbulences.

Contents

1 Introduction

In the previous talks we have acquire basic pictures of the developed turbulence. In particular, there are co-existing eddies of different size. As we inject energy into the fluid, the energy is cascaded into different scales. Self-similarity conditions are imposed upon them with constant energy flux rate, and consequently, we were able to proceed the physical argument assuming physical quantities have power-law dependence on the length scale.

We also discussed the competition between the convection term and dissipation term and argued the existence of the *inertial range* between the dissipation range, where ν starts taking over, and the injection range, where the energy is injected.

In this talk we are about to step forward, studying how energy spectrum, or spectrum of quadratic invariants in general, depends on the scales in various dimensions and possibly in the presence of magnetic field. The former fact influences the conserved charges in the theory whereas the latter promotes the hydrodynamic problem to the more difficult magnetohydrodynamic (MHD) problem. As we will see shortly, despite the intuition built in last talk based on 3 dimensional turbulence, as we vary the dimensions and turn on and off magnetic field, energy does not necessarily follow direct cascade. Instead, it may follow the inverse one and accumulates at large scale. We will also examine similar phenomenologies of other conserved quantities.

The talk is organized as follows. We first review the ideal dynamical equations governing the dynamics in both hydrodynamics and magnetohydrodynamics. Following the equations of motion we introduce the associated quadratic invariants, whose spectra will be studied qualitatively. Then we investigate how these invariant dissipate as dissipation terms are turned on, where we could already see the difference between 3 dimensional hydrodynamics and 2 dimensional one. In the latter case, selective decay will draw our attention. In the following section we perform a diagnosis in the absolute equilibrium limit, where we are able to define ensemble average to exactly calculate the equilibrium distribution of spectra, from which we infer the directions of cascade. Finally in section 5 we provide a back-in-the-envelope way of deriving the cascade spectrum including the famous K41 theory.

2 Equation of Motion

2.1 Dynamical Equation for Vorticity

Let us first recall the Euler equation for homogeneous and incompressible fluids, that

$$\Big(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\Big)\mathbf{v} = -\nabla p,\ \nabla\cdot\mathbf{v} = 0. \tag{2.1}$$

With the help of the vector identity $\mathbf{v}\times(\nabla\times\mathbf{v}) = \frac{1}{2}\nabla v^2 - \mathbf{v}\cdot\nabla\mathbf{v}$, it can be rewritten as

$$\frac{\partial\mathbf{v}}{\partial t} - \mathbf{v}\times\boldsymbol{\omega} = -\nabla(p + v^2/2), \tag{2.2}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{v}$ is the vorticity. In the first talk we mentioned it can be written in a p independent form by taking the curl of the whole equation. Taking the curl. $\nabla\times(\mathbf{v}\times\boldsymbol{\omega}) = (\boldsymbol{\omega}\cdot\nabla)\mathbf{v} - \boldsymbol{\omega}(\nabla\cdot\mathbf{v}) + \mathbf{v}(\nabla\cdot\boldsymbol{\omega}) - (\mathbf{v}\cdot\nabla)\boldsymbol{\omega}$ for incompressible fluid and $\nabla\times\nabla(p + v^2/2) = 0$. We obtain

$$\Big(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\Big)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{v}. \tag{2.3}$$

In the presence of viscosity, $\nu\nabla^2\boldsymbol{\omega}$ is added to the right-hand side

$$\Big(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\Big)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{v} + \nu\nabla^2\boldsymbol{\omega}. \tag{2.4}$$

2.2 Turning On Magnetic Fields

If we turn on a magnetic field, the change of the hydrodynamic equations is straightforward. As we commented in the first talk, it is a statement of force balance. As a consequence, we have to add the Lorenz force density to the force side

$$\delta\mathbf{f} = \mathbf{j}\times\mathbf{B},\ \nabla\times\mathbf{B} = \mathbf{j}. \tag{2.5}$$

This relation holds in non-relativistic case, where the displacement current give rise to correction of order $(v/c)^2$. We emphasize that there is no relation between ρ and $\mathbf{j}$ a priori, which depends specifically who the charge carriers are. To close the equations, we need the equation for $\mathbf{B}$ given by the Maxwell equation $-\frac{\partial}{\partial t}\mathbf{B} = \nabla\times\mathbf{E}$. Ohm's law and Galilean invariance implies

$$\frac{\partial}{\partial t}\mathbf{B} = \nabla\times(\mathbf{v}\times\mathbf{B}) + \frac{1}{\sigma}\nabla^2\mathbf{B} + \cdots\,, \tag{2.6}$$

where $(\cdots)$ includes higher order terms in gradient expansion such as hyperresistivity. The ideal and non-dissipative limit corresponds to $\sigma^{-1}\to 0$. For more details, see Ref. [4].

Turning on the magnetic field is more than adding an extra term and some equations to the problem. It introduces one extra scale, the gauge field, and may affect the scaling argument. The Alfvén wave velocity appears in the spectrum scaling in MHD turbulence.

3 Conserved Charges

For the moment we look at ideal fluids, in which $\sigma^{-1} = 0$ and $\nu = 0$. From the previous experience, we know a bunch of quantities are conserved such as energy, momentum, etc. In this section we list the conserved quantities of our interest in both hydrodynamics and MHD.

3.1 $d = 3$

In 3 dimensions, we can already list a few

$$E = \frac{1}{2}\int d^3x\; v^2 \tag{3.1}$$

$$\mathbf{P} = \int d^3x\, \mathbf{v} \tag{3.2}$$

$$\boldsymbol{\Omega} = \int d^3x\, \boldsymbol{\omega} \tag{3.3}$$

$$\int d\boldsymbol{\ell}\cdot\mathbf{v} = \int d\boldsymbol{a}\cdot\boldsymbol{\omega}, \tag{3.4}$$

where the conservation of the first 2 were verified in other talks, and that of the third one should be evident since $\boldsymbol{\omega}$ obeys almost the same equation as $\mathbf{v}$ does. The fourth one is the celebrated Kelvin's theorem [5].
There is one extra quadratic constant of motion, the *helicity*, defined as

$$H_\omega = \int d^3x\, \boldsymbol{\omega}\cdot\mathbf{v}, \tag{3.5}$$

which is related to the linking number between $\boldsymbol{\omega}$ vortex lines [6]. Its conservation can be shown as follows

$$\begin{aligned} \frac{\partial}{\partial t}\int d^3x\, \mathbf{v}\cdot\boldsymbol{\omega} &= \int d^3x\, \mathbf{v}\cdot(-(\mathbf{v}\cdot\nabla)\boldsymbol{\omega} + \boldsymbol{\omega}\cdot\nabla\mathbf{v}) \\ &+ \int d^3x\, \boldsymbol{\omega}\cdot(-\mathbf{v}(\cdot\nabla)\mathbf{v} - \nabla p) = \int d^3x\, \nabla\cdot(-\mathbf{v}(\mathbf{v}\cdot\boldsymbol{\omega}) + v^2\boldsymbol{\omega}/2 - p\boldsymbol{\omega}). \end{aligned} \tag{3.6}$$

Here I state without proof that E and H are the only 2 quadratic invariant integrals in 3 dimensional Euler fluid dynamics.
As $\mathbf{B}$ is turned on, the helicity is no longer conserved, while the energy should incorporate the magnetic contribution, that

$$E = \frac{1}{2}\int d^3x\,(v^2 + B^2). \tag{3.7}$$

Though we lose helicity, there are actually 2 more integral invariants, magnetic helicity and the cross helicty

$$H_B = \int d^3x\, \mathbf{A}\cdot\mathbf{B}, \tag{3.8}$$

$$K = \int d^3x\, \mathbf{v}\cdot\mathbf{B}. \tag{3.9}$$

We note that at the first glance H_B is not even well-defined because of gauge invariance. Indeed, there are some conditions required to properly define this quantity. In the following discuss, the results will be limited to those cases such as compactly supported fields or periodic boundary conditions. An example for which caution is in need is the field line reconnection problem.
H_B actually has a physical interpretation similar to H_ω, which is the linking number between flux tubes. Note that on that occasion this quantity is well-defined since no magnetic field line could penetrate the flux tube by definition.

3.2 $d = 2$

The story is a bit different in 2 dimensions. The energy is still a conserved quadratic integral. However, helicity $\int \mathbf{v} \cdot \boldsymbol{\omega}$ is identically 0 in 2 dimensions. To find the potential quadratic invariant, we note that $\boldsymbol{\omega} \to \omega$ is merely a pseudo-scalar, and there is no vorticity stretching term $\boldsymbol{\omega} \cdot \nabla \mathbf{v}$. In $\nu = 0$ limit, the equation ω obeys is

$$\Big(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\Big)\omega = 0. \tag{3.10}$$

Multiplying ω from the left and integrating over $\int d^2x$ yields

$$\frac{\partial}{\partial t} 2\Omega^{(2)} = \frac{\partial}{\partial t} \int d^2x\, \omega^2 = -\frac{1}{2} \int d^2x\, \nabla \cdot (\omega^2 \mathbf{v}) = 0. \tag{3.11}$$

In the same spirit, we can show there are infinitely many conserved charges $\Omega^{(n)}$, or any function $f(\omega)$. In fact, some decades ago, people thought there is no turbulence is purely 2 dimensions because of the enormous conserved charges.

The $n = 2$ charge, the *enstrophy*, is of particular interest. It plays the role of the second quadratic invariant in 2 dimensional Euler hydrodynamics.

Before we end this section, let us introduce the quadratic invariants in 2 dimensional MHD. In addition to the $B^2/2$ correction to energy, analogously, we have the cross-helicity and the mean-square magnetic potential

$$K = \int d^2x\, \mathbf{v} \cdot \mathbf{B} \tag{3.12}$$

$$A = \int d^2x\, A_z^2,\ \mathbf{B} = \nabla \times A_z \hat{\mathbf{z}}. \tag{3.13}$$

The conservation of these quantities can be confirmed using dynamical equations.

3.3 Dissipation and Selective Decay

Before examining the cascade phenomenology, let us first list the decay behavior of the ideal invariants. It also shed some lights to the difference between 2 dimensional and 3 dimensional physics.

Let us start with 3 dimensional hydrodynamic turbulence.

$$\dot{E} = -\nu \int d^3x\, \omega^2 \tag{3.14}$$

$$\dot{H}_\omega = -\nu \int d^3x\, \boldsymbol{\omega} \cdot (\nabla \times \boldsymbol{\omega}). \tag{3.15}$$

In 2 dimensions, the energy decays in the same way, while Ω is bounded.

$$\dot{E} = -\nu \int d^2x\, \omega^2 = -2\nu\Omega \tag{3.16}$$

$$\dot{\Omega} = -\nu \int d^2x\, (\nabla\omega)^2. \tag{3.17}$$

Before writing down the MHD cases, let us first point out the difference between the above 2 cases. In 3 dimensions, it is known that $\dot{E}$ may have finite off-set as we tune $\nu \to 0$. It is dubbed anomalous dissipation. It is because of the vortex stretching term $\boldsymbol{\omega} \cdot \nabla \mathbf{v}$. However, in 2 dimensions, there is no vortex stretching and Ω can only decrease as time evolves. The consequence is that at little ν, Ω decays much faster than E does. It is

called selective decay.
In MHD similar phenomenon arises. In 3 dimensions

$$\dot{E} = -\eta \int d^3x\, j^2 - \nu \int d^3x\, \omega^2 \tag{3.18}$$

$$\dot{K} = -(\eta + \nu) \int d^3x\, \boldsymbol{\omega} \cdot \mathbf{j} \tag{3.19}$$

$$\dot{H}_B = -\eta \int d^3x\, \mathbf{j} \cdot \mathbf{B}. \tag{3.20}$$

In 2 dimensional MHD, the first 2 identities can be duplicated, while the third one is replaced with the decay rate of A.

$$\dot{E} = -\eta \int d^2x\, j_z^2 - \nu \int d^2x\, \omega^2 \tag{3.21}$$

$$\dot{K} = -(\eta + \nu) \int d^2x\, \omega j_z \tag{3.22}$$

$$\dot{A} = -\eta \int d^2x\, B^2. \tag{3.23}$$

Now we see the selective decay happens for the pair (A, E), and in the following we indeed will see some analog between hydrodynamic turbulence $(E, \Omega) \leftrightarrow$ MHD turbulence (A, E).

4 Full Equilibrium Investigation

Now we have a rough picture of the problem to deal with. We are interested in how those conserved charges defined in the ideal system evolve, or cascade, in a fully developed turbulence, from the input length scale to a smaller or sometimes larger cutoff. Remind that the turbulent state is neither at equilibrium nor conservative. However, as we argued in previous talks, dissipation takes over at small scales. The nonlinear advection is the same regardless of dissipation. As a consequence, we can actually get some hints from equilibrium analysis.
To proceed, let us further include another simplifying assumption. We suppose there is a region in the system which is small compared to the entire flow but large enough to assume homogeneity and irrelevance of boundary conditions. Isotropy is not able to be assumed in all cases. In usual hydrodynamics, the main stream can be removed using Galilean transformation, while in MHD, magnetic field can not be eliminated that way.
With the above assumption, we can decompose the dynamical variables into Fourier modes. One may be concerned with the utility of Fourier representation in such a nonlinear system. The reason is that the cascade of ideal invariants between different scales can be interpreted as the cascade between different momenta.The main purpose here is to examine the weight of each Fourier mode instead of solving the equations of motion.
The analysis in this section is not much more than reviewing equilibrium thermodynamics. Given conserved integral invariants H_i, which assumes a Fourier decomposition $\sum_{\mathbf{k}} H_i(\mathbf{k})$, we construct a Gibbs ensemble

$$\rho = Z^{-1} \exp(-K) = Z^{-1} \exp\Big(- \sum_i \beta_i H_i \Big). \tag{4.1}$$

The equilibrium value $\langle \mathcal{O}(\mathbf{k}) \rangle$ is defined as an ensemble average.
We illustrate this technique for 2 dimensional hydrodynamic turbulence. Writing

$$\rho = Z^{-1} \exp(-\beta H - \alpha \Omega) \tag{4.2}$$

$$H = \frac{1}{2} \sum_{\mathbf{k}} \mathbf{v}_{-\mathbf{k}} \cdot \mathbf{v}_{\mathbf{k}} \tag{4.3}$$

$$\Omega = \frac{1}{2} \sum_{\mathbf{k}} \omega_{-\mathbf{k}} \omega_{\mathbf{k}} = \frac{1}{2} \sum_{\mathbf{k}} \epsilon^{mn} k^m v^n_{-\mathbf{k}} \epsilon^{ij} k^i v^j_{\mathbf{k}}. \tag{4.4}$$

Using $\nabla \cdot \mathbf{v} = 0$ it is possible to rewrite the exponent as

$$\frac{1}{2} \sum_{\mathbf{k}} (\beta + \alpha k^2) \mathbf{v}_{-\mathbf{k}} \cdot \mathbf{v}_{\mathbf{k}}. \tag{4.5}$$

Consequently, as we evaluate $E_k \propto k \langle \mathbf{v}_{-\mathbf{k}} \cdot \mathbf{v}_{\mathbf{k}} \rangle$, we obtain

$$E_k \propto \frac{k}{\beta + \alpha k^2}, \tag{4.6}$$

which is enhanced at low k. On the other hand, $\Omega_k \propto k^2 \langle \mathbf{v}_{-\mathbf{k}} \cdot \mathbf{v}_{\mathbf{k}} \rangle$, and therefore,

$$\Omega_k \propto \frac{k^3}{\beta + \alpha k^2}, \tag{4.7}$$

which can be enhanced at large k. Similar calculation can be carried out for 3 dimensional hydrodynamic and MHD. Nonetheless, the $\nabla \cdot \mathbf{v}$ needs to be implemented as a gauge fixing condition. In other words, the equation of motion should read

$$\left\langle v^j(\mathbf{k}) \frac{\delta K}{\delta v^i(\mathbf{k})} \right\rangle = P_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2} \tag{4.8}$$

instead of δ_{ij}. Solving the above equation for 3 dimensional hydrodynamic correlation function with,

$$K = \frac{1}{2} \sum_{\mathbf{k}} [\beta \mathbf{v}_{-\mathbf{k}} \cdot \mathbf{v}_{\mathbf{k}} + \alpha i \mathbf{v}_{-\mathbf{k}} \cdot (\mathbf{k} \times \mathbf{v}_{\mathbf{k}})]. \tag{4.9}$$

The equation of motion derived from this potential is

$$(\beta \delta_{im} + i \epsilon_{mli} k^l) \langle v^j(\mathbf{k}) v^m(-\mathbf{k}) \rangle = P_{ij}. \tag{4.10}$$

Inverting this matrix, we arrive at

$$\langle v^i(\mathbf{k}) v^j(-\mathbf{k}) \rangle = \frac{1}{\beta^2 - \alpha^2 k^2} (\beta P_{ij} + i \alpha \epsilon_{ilj} k^l), \tag{4.11}$$

following which

$$E_k \propto \frac{k^2}{(\beta^2 - \alpha^2 k^2)}, \ \beta^2 - \alpha^2 k^2 > 0. \tag{4.12}$$

$\alpha = 0$ corresponds to the non-helical flows and if we turn it on from 0, we see E_k is enhanced at large k in 3 dimensions. For more details, see Ref. [7].
Computation in MHD can be carried out in a similar manner with the inclusion of magnetic degrees of freedom.

Table 1. The ideal integral invariants and their cascade directions.

	3-D	2-D
MHD	E_k: Direct	E_k: Direct
	K_k: Direct	K_k: Direct
	H_{Bk}: Inverse	A_k: Inverse
Navier-Stokes	E_k: Direct	E_k: Inverse
	$H_{\omega k}$: Direct	Ω_k: Direct

We refer the interested readers to Ref. [2, 3]. What we are more interested in is the implication of equilibrium spectra. We emphasize again the absolute equilibrium is far from dissipative turbulence, but in the internal range the driving force is the same in both cases. Suppose we inject some amount of energy at an intermediate scale and the direction the it relaxes toward is expected to have a larger spectrum weight. That said, if the spectrum is peaked at large k at equilibrium, we anticipate a direct cascade. On the other hand, an inverse cascade is being looked forward to as the spectrum assumes maximum at small k. Following this logic, in 3 dimensions E_k follows direct cascade, whereas 2 dimensional E_k follows inverse cascade. In the same spirit, the cascade directions for all ideal invariants are listed in table 1.

5 Cascade Spectra

With some idea of the cascade direction in mind, in this section we are going discuss the cascade spectrum as a function of k. Let us first we recall the idea of inertial range. The basic idea is that is the conserved integral is pumped at an infrared scale, while the it is dissipated at ultraviolet scale. These 2 regime are characterized by k_{in} and k_{uv}. In between them, there is a range where the dynamics depends only on the non-linear term, but not on the external source or the dissipation. Similarly, there is another inertial range defined for inverse cascade.

$$\text{I1}:\ k_{\text{in}} \gg k \gg k_{\text{uv}} \tag{5.1}$$

$$\text{I2}:\ L^{-1} \gg k \gg k_{\text{in}}. \tag{5.2}$$

First we look at hydrodynamic turbulence. Based on the above assumption, the only dimensionful quantity in the inertial range is the momentum itself. Suppose we accept the idea here, we can actually give a back in the envelope argument for the scaling form of the energy spectrum $E(k)$. In 3 dimensions, the energy follows direct cascade. We thus assume the constant energy rate ϵ at every stage of cascade from k_{in} to k_{uv}. On the other hand, in 2 dimensions, it is enstrophy that follows direct cascade. Within the first inertial range, we instead assume the constant injection rate for Ω, ϵ_Ω. In units of L and T, it is clear that

$$[\epsilon_\Omega] = \frac{[\omega^2]}{[T]} = \frac{1}{T^3} \tag{5.3}$$

$$[\epsilon] = [v^2]/T = \frac{L^2}{T^3} \tag{5.4}$$

$$[E(k)] = \frac{[v^2]}{[k]} = \frac{L^3}{T^2}. \tag{5.5}$$

Therefore, using the inertial range picture, if we consider fixed ϵ, that means

$$\frac{L^3}{T^2} = [\epsilon]^\alpha [k]^\beta \Rightarrow \alpha = \frac{2}{3}, \beta = -\frac{5}{3} \tag{5.6}$$

and we arrive at the famous K41 result,

$$E(k) = C\epsilon^{3/2}k^{-5/3}. \tag{5.7}$$

In 3 dimensions, the constant C is known as the Kolmogorov constant. It seems to be an universal number around 1.6-1.7 from experiments over a board range of Reynolds number. Precise numerics done at Re$\simeq 500$ yields $C = 1.65 \pm 0.05$.
Looking at 2 dimensional flows, we impose fixed ϵ_Ω,

$$\frac{L^3}{T^2} = [\epsilon_\Omega]^{\alpha'}[k]^{\beta'} \Rightarrow \alpha' = \frac{2}{3}, \beta' = -3, \tag{5.8}$$

which implies a new relation

$$E(k) = C'{\epsilon_\Omega}^{2/3}k^{-3}. \tag{5.9}$$

$C' = 1.5$-1.7. Notice that in 2 dimensions we can also look at the inverse cascade of energy spectrum at large scales. Since the dimensional analysis does not rely on the direction of cascade, we know the spectrum follows the same scaling behavior with the modification $C \sim 6$.
We would like to propose another similar argument. Suppose there is a time scale τ after which the modes at order k no longer correlate with the assumption that the integral invariants' injection rate is proportional to this scale. For example, given ϵ or ϵ_Ω, dimensional analysis gives

$$\epsilon \sim \tau E_k^2 k^4 \tag{5.10}$$

$$\epsilon_\Omega \sim \tau E_k^2 k^6. \tag{5.11}$$

Therefore, the question left is to express τ in terms of other variables. In the K41 phenomenology, $\tau = \ell/\delta v = \ell/(\epsilon\ell)^{1/3} = \ell^{2/3}/\epsilon^{1/3}$, implying $E_k \propto \epsilon^{2/3}k^{-5/3}$. In 2 dimensional turbulence,

$$\epsilon_\Omega \sim \frac{1}{\tau}\frac{\delta v^2}{\ell^2} \sim \frac{1}{\tau^3} \Rightarrow \tau \sim \epsilon_\Omega^{-1/3}, \tag{5.12}$$

and hence $E_k \propto \epsilon_\Omega^{2/3}k^{-3}$. The reason that we introduce the second argument is that it can be easily applied to MHD, where the inertial range argument needs to be modified in the presence of the Alfvén wave, which provides another velocity scale v_A. Using the above argument and replacing $\tau^{-1} = v_A k$, we easily find

$$E_k = C_{\mathrm{IK}}(\epsilon v_A)^{1/2}k^{-3/2}, \tag{5.13}$$

where C_{IK} is the Iroshinikov-Kraichnan coefficient.
We comment that the spectrum can be obtained by other arguments such as the combination of self-similar spectrum and local interaction in momentum space, which may not be extremely illustrating here but points out some crucial idea behind the power-law spectrum. Interested readers may refer to Ref. [1]. Actually the breakdown of nonlocal interaction in 2 dimensional turbulence leads to logarithmic correction to the spectrum $(\log[k/k_{\mathrm{in}}])^{-1/3}$.
In previous discussions, we have seen different cascade phenomenology in various types of hydrodynamics. A natural question that has not been answered is that what leads to inverse cascade. We have understood that having several ideal invariants cannot be sufficient, since 3 dimensional hydrodynamic turbulence is a counterexample. It seems that we also require selective decay given those integral invariants, which means the decay rates of those integral invariants differ significantly. However, for the moment, I doubt it does close the sufficient condition since the absolute equilibrium diagnosis knows nothing about selective decay.

6 Summary

To summarize, in this talk we concentrate the cascade phenomenology in fully developed turbulence. More concretely, we consider quadratic integral invariants in the ideal system and study the scaling behavior of their Fourier modes and cascade directions. It may be worth mentioning that owing to the assumption of inertial range, we are able to attribute the driving force to the non-dissipative convection term and perform a lot of arguments in the ideal limit. After the diagnosis in the absolute equilibrium limit, we derive the energy spectrum in MHD turbulence, 3 dimensional hydrodynamic turbulence, and 2 dimensional hydrodynamic turbulence. To close, we refer to some numerical fact and known discrepancy with realities owing to the failure of starting assumptions. Though, as far as the author knew, the sufficient and necessary conditions for the existence of inverse cascade is still not settled, and it may be a question worth putting efforts into.

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