Quantum-classical mapping

- we will show that

d-dimensional quantum system $\longleftrightarrow$ $(d+1)$-dimensional statistical mechanics system

- For quantum Hamiltonian $\hat{H}$ at temperature T,

$$Z = \mathrm{Tr}\, e^{-\beta \hat{H}} = \sum_m \langle m | e^{-\beta \hat{H}} | m \rangle$$ where $\{|m\rangle\}$ is some basis.

Break up $e^{-\beta \hat{H}} = e^{-d\tau \hat{H}} e^{-d\tau \hat{H}} \cdots e^{-\delta\tau \hat{H}}$ where $N\delta\tau = \beta$.

There are $O(\delta\tau^2)$ terms that we are ignoring for small $\delta\tau$ later on.

Insert resolutions of identity to get

$$Z = \sum_{m_0, m_1, \ldots, m_{N-1}} \langle m_0 | e^{-d\tau \hat{H}} | m_{N-1} \rangle \langle m_{N-1} | e^{-d\tau \hat{H}} | m_{N-2} \rangle \langle m_{N-2} | \cdots | m_1 \rangle \langle m_1 | e^{-d\tau \hat{H}} | m_0 \rangle$$

For the transverse field quantum Ising model in 1D

$$\hat{H} = \underbrace{-J \sum_{\langle ij \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z}_{H_z} \underbrace{- \Gamma \sum_i \hat{\sigma}_i^x}_{H_x}$$

Work in the basis of $\hat{\sigma}_i^z$'s, so $|m\rangle = |S_z(1), S_z(2), \ldots \rangle = |\{S_z(i)\}\rangle$

with $S_z = \pm 1$.

For a single time step,

$$\langle m_{\tau+d\tau} | e^{-d\tau \hat{H}} | m_\tau \rangle \simeq \langle m_{\tau+d\tau} | e^{-d\tau H_x} e^{-d\tau H_z} | m_\tau \rangle$$

$$= e^{-d\tau E_z(\{S_z(i,\tau)\})} \langle m_{\tau+d\tau} | e^{-d\tau H_x} | m_\tau \rangle$$

$$= e^{-d\tau E_z(\{S_z(i,\tau)\})} \prod_{i \leftarrow \text{run over sites}} \langle S_{z,i}(\tau+d\tau) | e^{\Gamma \hat{\sigma}_i^x} | S_{z,i}(\tau) \rangle$$

For each site,

$$\langle S_z(\tau+d\tau) | e^{d\tau \Gamma \hat{\sigma}^x} | S_z(\tau) \rangle$$

$$= \sum_{S_x = \pm 1} \langle S_z(\tau+d\tau) | e^{d\tau \Gamma \hat{\sigma}^x} | S_x \rangle \langle S_x | S_z(\tau) \rangle$$

$$= \sum_{S_x = \pm 1} e^{d\tau \Gamma S_x} \langle S_z(\tau+d\tau) | S_x \rangle \langle S_x | S_z(\tau) \rangle$$

One can show that $\langle S_x | S_z \rangle = \frac{1}{\sqrt{2}} e^{i\pi \frac{1-S_x}{2} \cdot \frac{1-S_z}{2}}$

Note that $\langle S_x | S_z \rangle$ is real, so $\langle S_z | S_x \rangle = \frac{1}{\sqrt{2}} e^{i\pi \frac{1-S_x}{2} \frac{1-S_z}{2}}$ as well.

So, for the single spin,

$$\underbrace{\langle S_z'|}_{S_z(\tau+\delta\tau)} e^{\delta\tau\Gamma\hat{\sigma}^x}|S_z\rangle = \sum_{S_x=\pm1} e^{\delta\tau\Gamma S_x}\,\frac{1}{2}\, e^{i\pi\frac{1-S_x}{2}\left(\frac{1-S_z'}{2}+\frac{1-S_z}{2}\right)}$$

$$= \frac{1}{2}\Big(e^{\delta\tau\Gamma} + e^{-\delta\tau\Gamma}\underbrace{e^{i\pi\left(\frac{1-S_z'}{2}\right)}}_{S_z'}\underbrace{e^{i\pi\left(\frac{1-S_z}{2}\right)}}_{S_z}\Big)$$

$$= \frac{1}{2}\left(e^{\delta\tau\Gamma} + e^{-\delta\tau\Gamma}S_z'S_z\right) \quad (1)$$

Try to put this into the form

$$\langle S_z'|e^{\delta\tau\Gamma\hat{\sigma}^x}|S_z\rangle = c\,e^{J_\tau S_z'S_z} = c\left(\cosh J_\tau + S_z'S_z\sinh J_\tau\right) \quad (2)$$

Comparing (1) and (2) gives (divide the coeff of $S_z'S_z$

$$e^{-2\delta\tau\Gamma} = \tanh J_\tau$$

So, for the transverse field quantum Ising model,

$$Z = \sum_{\{S_z(i,\tau_0);\,S_z(i,\tau_1),\dots,S_z(i,\tau_{N-1})\}} \overset{N\text{ factors}\ \to}{c}^{\,NL\ \leftarrow\ \#\text{ of sites}}$$

$$\exp\Big[\underbrace{J\,\delta\tau}_{\equiv J_{spat}}\sum_{\tau,\langle ij\rangle}S_z(i,\tau)S_z(j,\tau) + \sum_{\tau,i}J_\tau\,S_{z,i}(\tau+\delta\tau)\,S_z(\tau)\Big]$$

$J_\tau = \tanh^{-1}(e^{-2\delta\tau\Gamma})$

$J_{spat} = J\delta\tau$

} length β. For $T\to0$, $\beta\to\infty$, infinite lattice.

- d dim quantum Ising in transverse field $\longleftrightarrow$ d+1 dim classical Ising with size $\beta = 1/T$ (because $\tau = 0, d\tau, \dots, N\delta\tau = \beta$
- $T=0$ quantum means classical Ising ∞ size in (d+1) dim
- For small $\delta\tau\to0$, J_{spat} small, J_τ large, anisotropic Ising.
- Isotropic classical Ising

$$J_{spat} = J_\tau \equiv K, \text{ so } K = \tanh^{-1}\left[e^{-2K\Gamma/J}\right]$$

$$\Rightarrow \Gamma/J = -\frac{1}{2K}\ln(\tanh K)$$

$$Z = \sum \exp\Big[K\Big(\sum_{\tau,\langle ij\rangle}S_z(i,\tau)S_z(j,\tau) + \sum_{\tau,i}S_{z,i}(\tau+\delta\tau)S_z(\tau)\Big)\Big]$$

so K is the effective inverse temp for classical model.

d dim Quantum	(d+1) dim Classical
large Γ/J quantum disordered phase	small $K = e^{-2K\Gamma/J}$ high-temp phase (disordered)
small Γ/J quantum ordered phase	large K low-temp (ordered) phase for $d+1 \geq 2$
quantum crit pt	classical Ising critical point.

e.g. 0-dim quantum ising is just single spin $H = -\Gamma\sigma^x$

G.S. $|\rightarrow\rangle$, $E = -\Gamma$

1st excited $|\leftarrow\rangle$, $E = \Gamma$ $\Rightarrow Z = e^{\beta\Gamma} + e^{-\beta\Gamma}$

∴ No phase transition, just a gapped qm system.

The equivalent 1d classical Ising with

$$K = J_I = \tanh^{-1}(e^{-2\delta\tau\Gamma})$$

is disordered for all $T > 0$.

Self-duality of isotropic Ising model (Classical 2D)

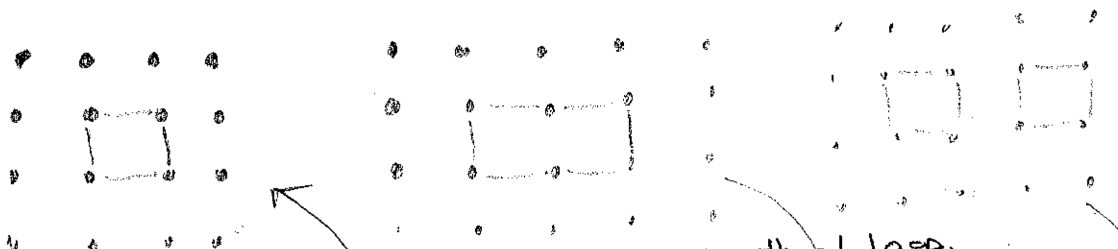
High Temp: $Z(K) = \sum_{\sigma_1 \cdots \sigma_N} e^{K \sum_{\langle ij \rangle} \sigma_i \sigma_j}$ $\quad K = \beta J$. is small.

Note that $\exp(K\sigma) = \cosh K + \sigma \sinh K$.

$$\Rightarrow Z(K) = \sum_{\sigma_1 \cdots \sigma_N} \prod_{\langle ij \rangle} (\cosh K + \sigma_i \sigma_j \sinh K) \qquad \sum_{\sigma_i} \sigma_i^{odd} = 0$$

$$= (\cosh K)^{\overset{2N \text{ links}}{2N}} \sum_{\sigma_1 \cdots \sigma_N} \prod_{\langle ij \rangle} (1 + \sigma_i \sigma_j \tanh K). \qquad \sum_{\sigma_i} \sigma_i^{even} = 2$$

$\prod_{\langle ij \rangle} (1 + \sigma_i \sigma_j \tanh K)$ is a sum of terms where each term has a 1 or $\sigma_i \sigma_j \tanh K$ on each link. If a term has odd number of σ_i for site i, $\sum_{\sigma_i}$ gives zero. So, need even number of σ_i's on each site. The nonzero terms correspond to closed loops

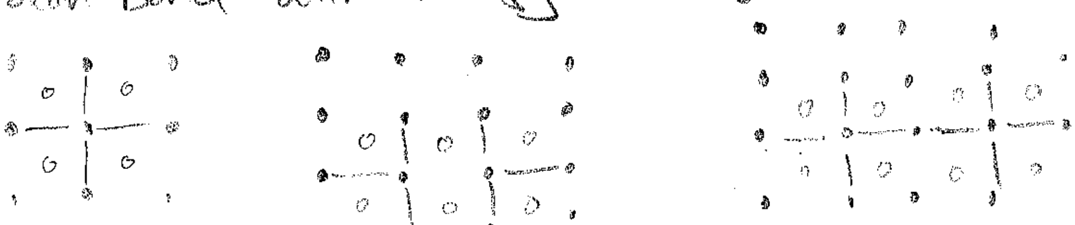


length of loop.

$$Z(K)(\cosh K)^{-2N} 2^{-N} = \sum_{\substack{\text{closed}\\ \text{loops}}} (\tanh K)^L$$

$$= 1 + \underset{\text{N possible ones}}{N} (\tanh K)^4 + 2N (\tanh K)^6 + \tfrac{1}{2} N(N-5)(\tanh K)^8$$

Low Temp: Choose $T=0$ ground state to be all up. Expand in number of spin flips. Each spin flip corresponds to a broken bond with energy cost $2J$.



$$Z(K) e^{-2NK} = \sum_{\substack{\text{droplets of}\\ \text{spin}}} e^{-2K \times \text{boundary of droplets}}$$

$$= 1 + N e^{-8K} + 2N e^{-12K} + \tfrac{1}{2} N(N-5) e^{-16K}$$

$$-\beta f = \frac{\ln Z}{N} = 2K + \frac{1}{N} \ln\left(1 + N e^{-8K} + 2N e^{-12K} + \frac{N(N-5)}{2} e^{-16K} + \ldots\right)$$

$$= 2K + e^{-8K} + 2e^{-12K} - \tfrac{5}{2} e^{-16K} + \ldots$$

N^2 terms from disconnected diagrams ...

The series are the same if we identify

$$\underset{\text{high } T}{\tanh K} = \underset{\text{low } T}{e^{-2K^*}}$$

$$\Rightarrow \quad \frac{Z(K^*)}{(e^{2K^*})^N} = \frac{Z(K)}{2^N(\cosh^2 K)^N}$$

Manipulating the hyperbolic functions give

$$\sinh(2K)\sinh(2K^*) = 1, \qquad \frac{Z(K^*)}{\sinh^{N/2}(2K^*)} = \frac{Z(K)}{\sinh^{N/2}(2K)}$$

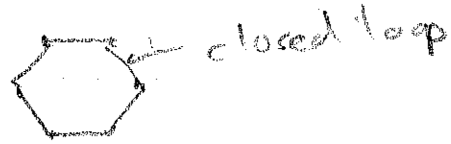
Krämers-Wannier duality.

If the critical point is unique, then it must occur at $K = K^*$ (Note that we made expansions, not approximations, so the series is exact). Then, $\sinh^2(2K_c) = 1 \Rightarrow e^{2K_c} = \sqrt{2} + 1$.

- Unfortunately, does not give critical exponents.
- for finite lattice, have sum of finite terms, hence convergent, hence duality is exact at any T.
- Also works on triangular and hexagonal lattices.

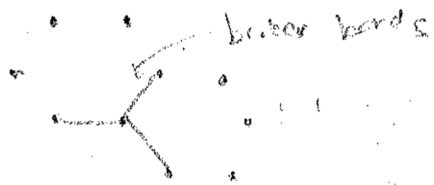
Low T triangle

High T hexagon



High T triangle

Low T hexagon



Self-duality of quantum Ising

$\hat{H} = -\sum_m \hat{\sigma}_1(m) - \lambda \sum_m \hat{\sigma}_3(m+1)\hat{\sigma}_3(m)$ drop hats for convenience.

$\frac{1}{\lambda} \sim$ temperature because large λ = ordered $\langle \sigma_3 \rangle \neq 0$

small λ = disordered $\langle \sigma_3 \rangle = 0$.

Define operators on dual lattice

$\mu_1(n) = \sigma_3(n+1)\sigma_3(n)$

$\mu_3(n) = \prod_{m<n} \sigma_1(m)$

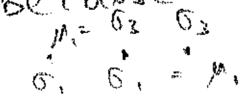
- μ_1 checks if adjacent spins are aligned
  μ_3 flips all spins to the left of n.
- Can be shown that the dual operators satisfy the same Pauli spin algebra as σ_1 and σ_3, i.e. $\sigma_1(n)\sigma_3(n) = -\sigma_3(n)\sigma_1(n)$

$\sigma_1^2(n) = \sigma_3^2(n) = 1$

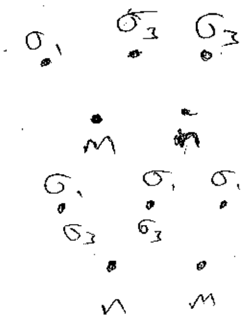
$[\sigma_1(n), \sigma_3(m)] = 0$ if $n \neq m$.

Can check μ satisfy this as well.

for e.g. $\mu_1(n)\mu_3(n) = -\mu_3(n)\mu_1(n)$ because there is an overlapping σ_x, σ_z.



- For $n \neq m$, $\mu_1(n)\mu_3(m) = \mu_3(m)\mu_1(n)$
  because only even number of μ's overlap.



- Note that $\sigma_1(m) = \mu_3(m+1)\mu_3(m)$

so $H = -\sum_n \mu_3(n)\mu_3(n+1) - \lambda \sum_n \mu_1(n)$

$= \lambda \left[-\sum_n \mu_1(n) - \lambda^{-1} \sum_n \mu_3(n)\mu_3(n+1) \right]$

So, $H(\sigma;\lambda) = \lambda H(\mu;\lambda^{-1})$

$\Rightarrow E(\lambda) = \lambda E(\lambda^{-1})$

Mass gap vanishes at critical point. If critical point is unique, then if it vanishes at λ, it must also vanish at λ^{-1} as well, and $\lambda_c = \lambda_c^{-1} \Rightarrow \lambda_c = 1$.

Next, we solve the 1D quantum Ising exactly to show that mass gap $G(\lambda) = 2|1-\lambda|$

Exact solution of 1D quantum Ising (Jordan-Wigner)

$H = -\sum_n \hat{\sigma}_3(n) - \lambda \sum_n \hat{\sigma}_1(n)\hat{\sigma}_1(n+1)$

Turn this into a "free" fermion.

Define raising, lowering: $\sigma^+(n) = \frac{1}{2}[\sigma_1(n) + i\sigma_2(n)]$

$\sigma^-(n) = \frac{1}{2}[\sigma_1(n) - i\sigma_2(n)]$

Label lattice sites $n = -N, -N+1, \ldots, N$.

$$C(n) = \prod_{j=-N}^{n-1} \exp[i\pi\sigma^+(j)\sigma^-(j)]\sigma^-(n)$$

$$C^+(n) = \sigma^+(n)\prod_{j=-N}^{n-1} \exp[-i\pi\sigma^+(j)\sigma^-(j)].$$

Idea: σ^+, σ^- anticommute on same site and square to zero. Want to form a string of operators so they anticommute everywhere, like fermions. Observe that

$$\sigma^-(n)\sigma^+(n) = \frac{1}{2}[\sigma_1(n) - i\sigma_2(n)]$$

$$\sigma^+(n)\sigma^-(n) = \frac{1}{2}[1 + \sigma_3(n)] \qquad (4.86)$$

$\exp[i\frac{\pi}{2}\sigma_3] = i\sigma_3$ check for $\sigma_3 = \pm 1$.

Allow us to write $C(n) = \prod_{j=-N}^{n-1}[-\sigma_3(j)]\sigma^-(n)$

$$C^+(n) = \sigma^+(n)\prod_{j=-N}^{n-1}[-\sigma_3(j)]$$

Easy to verify the usual anticommutation relations.

$\{c(n), c^+(m)\} = \delta_{nm}$ $\qquad \{c(n), c(m)\} = 0.$

For eg, $c(n)c^+(n) + c^+(n)c(n) = \sigma^-(n)\sigma^+(n) + \sigma^+(n)\sigma^-(n) = 1.$

For $m < n$, $c(m)c^+(n) + c^+(n)c(m) = \sigma^-(m)\prod_{j=m}^{n-1}[-\sigma_3(j)]\sigma^+(n) - \sigma^+(n)\prod_{j=m}^{n-1}[-\sigma_3(j)]\sigma^-(m)$

pick up − $\qquad$ no − $\qquad$ no −

$= 0$ $\qquad$ using $\sigma^-(m)\sigma_3(m) = -\sigma_3(m)\sigma^-(m)$

- Write H in terms of c, c^+. Using (4.86),

$\sigma_3^{(n)} = 2\sigma^+(n)\sigma^-(n) - 1 = 2c^+(n)c(n) - 1$

Note that $c^+(n)c(n+1) = \sigma^+(n)[-\sigma_3(n)]\sigma^-(n+1)$

But $\sigma^+(n)\sigma_3(n) = -\sigma^+(n)$

So $c^+(n)c(n+1) = \sigma^+(n)\sigma^-(n+1).$

Can similarly show that: $c(n)c^+(n+1) = -\sigma^-(n)\sigma^+(n+1)$

$c^+(n)c^+(n+1) = \sigma^+(n)\sigma^+(n+1)$

$c(n)c(n+1) = -\sigma^-(n)\sigma^-(n+1)$

Our coupling term becomes

$$\sigma_1(n)\sigma_1(n+1) = [\sigma^+(n) + \sigma^-(n)][\sigma^+(n+1) + \sigma^-(n+1)]$$
$$= [c^+(n) - c(n)][c^+(n+1) + c(n+1)]$$

We get a quadratic fermionic Hamiltonian.

$$H = -2\sum_n c^+(n)c(n) - \lambda \sum_n [c^+(n) - c(n)][c^+(n+1) + c(n+1)]$$

Have translational inv, so take Fourier transform to diagonalize.

$$c(n) = \sqrt{\frac{1}{2N+1}} \sum_k e^{-ikn} a_k \quad , k = 0, \pm\frac{2\pi}{2N+1}, \pm\frac{4\pi}{2N+1}, \ldots \pm\frac{2\pi N}{2N+1}$$

can show that a_k's are fermionic i.e $\{a_{k'}^+, a_k\} = \delta_{k',k}$

$$\{a_{k'}^+, a_k^+\} = \{a_{k'}, a_k\} = 0$$

$$\sum_n c^+(n)c^+(n+1) = \frac{1}{2N+1} \sum_n \sum_k \sum_{k'} e^{ikn} e^{ik'(n+1)} a_k^+ a_{k'}^+$$

skip to this

$$= \sum_{k,k'} e^{ik'} \delta_{k,-k'} a_k^+ a_{k'}^+$$
$$= \sum_k e^{-ik} a_k^+ a_{-k}^+$$

FT other terms as well. Get

$$H = -2\sum_k a_k^+ a_k - \lambda \sum_k (e^{-ik} a_k^+ a_{-k}^+ + e^{-ik} a_k^+ a_k + e^{ik} a_k^+ a_k - e^{ik} a_k a_{-k})$$
$$= -2\sum_k (1+\lambda\cos k) a_k^+ a_k - \lambda\sum_k (e^{-ik} a_k^+ a_{-k}^+ - e^{ik} a_k a_{-k})$$

Transform a_k, a_k^+ into new set of fermionic operators

$\eta_k = u_k a_k + i v_k a_{-k}^+ \qquad \eta_{-k} = u_k a_{-k} - i v_k a_k^+$

$\eta_k^+ = u_k a_k^+ - i v_k a_{-k} \qquad \eta_{-k}^+ = u_k a_{-k}^+ + i v_k a_k \qquad$ for $k > 0$.

Then,

$$\{\eta_{k'}, \eta_k^+\} = \delta_{k',k} \,,\ \{\eta_{k'}, \eta_k\} = \{\eta_{k'}^+, \eta_k^+\} = 0 \Rightarrow u_k^2 + v_k^2 = 1. \quad (4.111)$$

Write H in terms of these operators

$$H = \sum_{k>0} [-2(1+\lambda\cos k)(u_k^2 - v_k^2) + 4\lambda \sin k\, u_k v_k](\eta_k^+\eta_k + \eta_{-k}^+\eta_{-k})$$
$$+ \sum_{k>0} [4i(1+\lambda\cos k)u_k v_k + 2i\lambda \sin k\,(u_k^2 - v_k^2)](\eta_k^+\eta_{-k}^+ + \eta_k\eta_{-k})$$

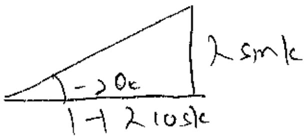
To make this diagonal, i.e only $\eta^+\eta$ term, need

$$4(1+\lambda\cos k)u_k v_k + 2\lambda\sin k\,(u_k^2 - v_k^2) = 0. \quad (4.114)$$

By (4.111), write $u_k = \cos\theta_k$, $v_k = \sin\theta_k$

Then, (4.114) becomes $2(1+\lambda\cos k)\sin 2\theta_k + 2\lambda\sin k\cos 2\theta_k = 0$.

$$\Rightarrow \tan 2\theta_k = -\frac{\lambda\sin k}{1+\lambda\cos k}$$



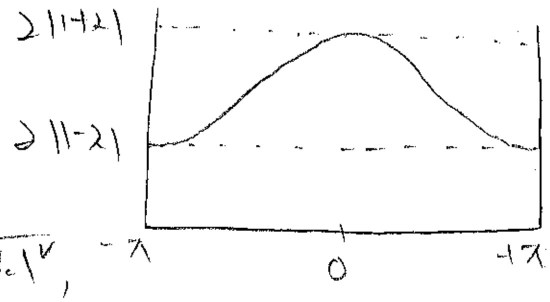
Can choose sign such that

$$\sin(2\theta_k) = \frac{\lambda \sin k}{\sqrt{1+2\lambda\cos k+\lambda^2}}$$

$$\cos(2\theta_k) = -\frac{1+\lambda\cos k}{\sqrt{1+2\lambda\cos k+\lambda^2}}$$

$$\Rightarrow \quad H = 2\sum_k \sqrt{1+2\lambda\cos k+\lambda^2}\; \eta_k^\dagger \eta_k + \text{const.}$$

$$\Lambda_k = 2\sqrt{1+2\lambda\cos k+\lambda^2}$$



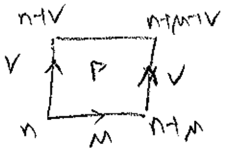
Minimum Λ_k at $k=\pm\pi$,
where $\Lambda_{\pm\pi} = 2|1-\lambda|$ gap.

Since $\Lambda \sim \frac{1}{\xi}$, and $\xi \sim \frac{1}{|\lambda-\lambda_c|^\nu}$,

$\nu = 1$.

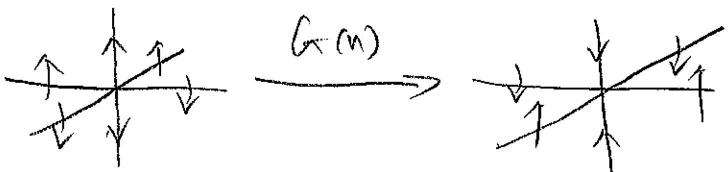
Ising gauge theory: D dimensional lattice with spins on links.

$$H = -J \sum_{n,\mu\nu} \sigma_3(n,\mu)\sigma_3(n+\mu,\nu)\sigma_3(n+\mu+\nu,-\mu)\sigma_3(n+\nu,-\nu)$$



$$= -J \sum_{P} \sigma_3\sigma_3\sigma_3\sigma_3$$

Gauge transform $G(n)$ = flip all spins on links of n.



Since the links of each site can border a plaquette two or zero times, we see that applying G does not affect any of the plaquettes.

$\left(\prod_i \sigma_3(i)\right)^2 = 1 \Rightarrow \prod_i \sigma_3(i) = \pm 1$. "$-1$" is frustrated.

Elitzur's theorem: local symmetry cannot break spontaneously. In this context $\langle \sigma_3 \rangle$ vanishes identically at all T, so we cannot use it as an order parameter.

Proof: To see if spontaneous magnetization is possible, apply external magnetic field, which results in $h \sum_{n,\mu} \sigma_3(n,\mu)$. Take $h \to 0$. If $\lim_{h\to 0} \langle \sigma_3(m,\nu)\rangle \neq 0$, then system is magnetized. If $\langle \sigma_3(m,\nu)\rangle \neq 0$, then not only have spontaneous breaking of global up $\rightleftharpoons$ down symm, but also the local symm as well. Consider

$$\langle \sigma_3(n,\nu)\rangle_h = \frac{\sum_{\text{spin config}} \sigma_3(n,\nu) \exp\{\beta \sum \sigma_3\sigma_3\sigma_3\sigma_3 + h \sum \sigma_3\}}{\sum_{\text{spin config}} \exp\{\beta \sum \sigma_3\sigma_3\sigma_3\sigma_3 + h \sum \sigma_3\}}.$$

Consider a local gauge transform at site n, $G(n)$. Denote the links coming from site n by $\{\lambda_n\}$.

Plaquette terms invariant by external field term becomes

$$h \sum \sigma_3 = h \sum \sigma_3' - h \sum \delta\sigma_3$$

where σ_3' is transformed spin and $\sigma_3'(\lambda_n) = -\sigma_3(\lambda_n)$

$$\delta\sigma_3(\lambda_n) = \sigma_3'(\lambda_n) - \sigma_3(\lambda_n) = -2\sigma_3(\lambda_n) \quad \text{if } \lambda_n \in \{\lambda_n\}.$$

$$\delta\sigma_3(\lambda) = 0 \quad \text{if } \lambda \notin \{\lambda_n\}.$$

Change of variables $\sigma_3 \to \sigma_3'$

$$\langle \sigma_3(n,\nu)\rangle_h = -\frac{\sum \sigma_3'(n,\nu)\exp\{\beta\sum\sigma_3'\sigma_3'\sigma_3'\sigma_3' + h\sum\sigma_3' - h\sum d\sigma_3\}}{Z}$$

$$= \langle -\sigma_3(n,\nu)\exp\{-h\sum_{\{\lambda_n\}} d\sigma_3\}\rangle_h$$

Now, we can get an upper bound

$$|\langle\sigma_3(n,\nu)\rangle_h - \langle -\sigma_3(n,\nu)\rangle_h|$$

$$= |\langle -\sigma_3(n,\nu)[\exp\{-h\sum_{\{\lambda_n\}} d\sigma_3\} - 1]\rangle_h$$

$$= |\langle -\sigma_3[\exp\{2h\sum_{\{\lambda_n\}}\sigma_3(\lambda_n)\} - 1]\rangle_h|$$

takes larges possible value when $\sigma_3(\lambda_n) = 1$

$$\le |e^{4dh} - 1|\,|\langle\sigma_3(n,\nu)\rangle_h| \xrightarrow{h\to 0} 0$$

$$\Rightarrow \langle\sigma_3(n,\nu)\rangle_{h\to 0} = \langle -\sigma_3(n,\nu)\rangle_{h\to 0}.$$

$$\Rightarrow \langle\sigma_3(n,\nu)\rangle = 0.$$

- Since we have no local order parameter, how to distinguish phases?
- Wegner suggest to look at spatial dependence of correlation functions.
- Inspired by 2D XY model, phase transition without spontaneous magnetization.

 low Temp: $\langle s(0)\cdot s(n)\rangle \sim |n|^{-kT/2\pi J}$

 high Temp: $\langle s(0)\cdot s(n)\rangle \sim \exp\left(-\frac{|n|}{\xi(T)}\right)$

 Diff behavior $\Rightarrow$ must have phase transition.
- Consider a gauge-invariant correlation function

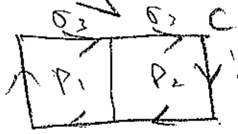
$$\langle \prod_{\lambda\in C}\sigma_3(\lambda)\rangle \quad \text{where } C \text{ is closed loop.}$$

High T: $\exp(\beta\,\sigma_3\sigma_3\sigma_3\sigma_3) = \cosh\beta + \sigma_3\sigma_3\sigma_3\sigma_3\sinh\beta$
$= (1 + \sigma_3\sigma_3\sigma_3\sigma_3\tanh\beta)\cosh\beta.$

$$\langle\prod_{\lambda\in C}\sigma_3(\lambda)\rangle = \frac{\sum_{\text{spin config}}\left[\prod_{\text{plaquettes}}(1+\sigma_3\sigma_3\sigma_3\sigma_3\tanh\beta)\right]\prod_C\sigma_3}{\sum\prod(1+\sigma_3\sigma_3\sigma_3\sigma_3\tanh\beta)}$$

only surviving term is $\sum(\prod 1) = 2^N$ to lowest order.

Numerator: 1^{st} non vanishing contribution is a minimal surface bounded by C.

e.g. 

: all σ's in this term square to 1.

So, $\langle T_c^! \sigma_3 \rangle = (\tanh\beta)^{N_c}$ where N_c is # of square in minimal surface.

$= \exp\{\ln(\tanh\beta) A\} + \ldots$

complicated function of β

To higher orders, find $\langle T_c^! \sigma_3 \rangle = \exp\{-f(\beta) A\}$

$\Rightarrow$ Area law.

Low temp and $d > 2$

For each set of gauge equivalent configurations, we can pick a representative spin configuration. So, when summing over spin configs, just sum over representative spin configs and multiply by same multiplicity factor.

- Aside: What is this multiplicity factor? Given a reference spin config $|\{\sigma_3\}\rangle$, any other config that is gauge equivalent to this can be expressed as $G^{n_1}(1) \ldots G^{n_N}(N) \,|\{\sigma_3\}\rangle$ where $n_1, \ldots n_N = 0, 1$.
  (N: # of sites)
  So, for all equivalence classes of spin configs, the multiplicative factor is the same, i.e. 2^N.

- This multiplicity factor occurs in both numerator and denominator, so it cancels out.
- Represent the $T=0$ spin config as all spin up, and expand in # of flipped spins.

$$\langle T_c^! \sigma_3 \rangle = \frac{\sum \prod \sigma_3 \exp\{\beta \sum \sigma_3\sigma_3\sigma_3\sigma_3\}}{Z}$$

1^{st} term $= T_c^! \sigma_3 = 1$ for all spin up.

<u>1 spin flip</u>: For $\lambda \gg$, $2(d-1)$ plaquettes frustrated ← each neighboring plaquette can be thought of as a link on a dual lattice with 1 less dimension.

Each frustrated plaquette has relative energy 2, so 1 spin flip has relative energy $4(d-1)$. If N links on lattice and L links in contour,

$$\langle \prod_c \sigma_3 \rangle = \frac{1 + \overbrace{(N-L)e^{-4(d-1)\beta}}^{\text{spin flip off lattice}} - \overbrace{L\, e^{-4(d-1)\beta}}^{\text{spin flip on lattice}}}{1 + N e^{-4(d-1)\beta}}$$

For n spin flips, approximating them as completely independent should be valid for $\lambda \gg n$, should get $\frac{1}{n!} N^n \exp[-4n(d-1)\beta]$

Then,

$$\langle \prod_c \sigma_3 \rangle = \frac{\sum_n \frac{1}{n!}(N-2L)^n \exp[-4(d-1)\beta]^n}{\sum_n \frac{1}{n!} N^n \exp[-4(d-1)\beta]^n}$$

$$= \frac{\exp[(N-2L)\exp[-4(d-1)\beta]]}{\exp[N \exp[-4(d-1)\beta]]}$$

$$= \exp[-2L\, e^{-4(d-1)\beta}]$$

completely independent spin flips,

If we don't assume completely independent spin flips,

$$\langle \prod_c \sigma_3 \rangle \propto \exp[-h(\beta) L] \quad \text{Perimeter law.}$$

So, high temp ⇒ Area law

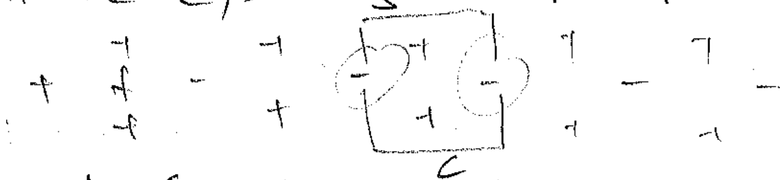
low temp ⇒ Perimeter law.

- This low temp argument does not apply to 2D classical Ising gauge theory. Consider a line of spins flipped

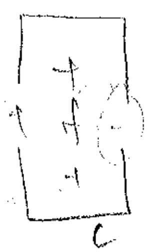
- Only two frustrated plaquettes whatever the length of the string, similar to how a domain in 1D classical Ising only pays energy penalty at its end, independent of domain length.
- This tends to disorder the system at any nonzero T, leading to area law even in low T.

- To see this, calculate gauge invariant correlation functions, only taking into account spin configurations in which one end of the line extends to infinity, leaving a "free" frustration.
- Suppose one frustration near contour C.

If f outside C, string of flip spins affect even spins on C.



If f inside C, string of flipped spins affect odd spins on C.



giving -1.

If N_c is number of links enclosed in C, then

$$\langle \prod_c \sigma_3 \rangle = \frac{1 + (M - N_c) e^{-2\beta} - N_c e^{-2\beta}}{1 + M e^{-2\beta}}$$

If we treat multiple frustrations as being independent, summing over their number gives

$$\langle \prod_c \sigma_3 \rangle \approx \exp(-2 e^{-2\beta} N_c) = \exp(-e^{-2\beta} A)$$

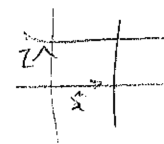
- Area law, even for low temp.
- $\langle \prod_c \sigma_3 \rangle$ does not detect any phase transition because there isn't.
- Follows from the duality

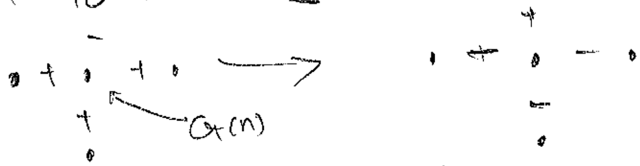
$$\begin{pmatrix} \text{2-dim} \\ \text{Ising gauge theory} \end{pmatrix} = \begin{pmatrix} \text{One-d classical} \\ \text{Ising} \end{pmatrix}$$

Two-dim Ising gauge theory = 1d classical Ising

Fix "temporal gauge", i.e. $\sigma_3(n, \hat{\tau}) = 1$ ← temporal links.

Given a flipped temporal link, apply successive $G(n)$ to move it to infinity.





- $H = -J \sum_{t} \sigma_3(n, \hat{x})\, \sigma_3(n+\hat{\tau}, \hat{x})$

- No coupling in spacial direction, only temporal direction. Becomes a classical Ising 1D model, which is disordered for all $T \neq 0$.

- Calculate $\langle T_c^{!}\, \sigma_3 \rangle$.

$$T_c^{!}\, \sigma_3 = \sigma_3(0,0,\hat{x})\, \sigma_3(0,1,\hat{x}) \cdots \sigma_3(0,R,\hat{x}) \times \sigma_3(T,0,\hat{x})\, \sigma_3(T,1,\hat{x}) \cdots \sigma_3(T,R,\hat{x})$$

- These correlation functions are short ranged

$$\langle \sigma_3(T,0,\hat{x})\, \sigma_3(0,0,\hat{x}) \rangle_{\substack{\text{1-d} \\ \text{Ising}}} \underset{T\to\infty}{\sim} \exp(-T/\xi)$$

so, $\langle T_c^{!}\, \sigma_3 \rangle \sim \left[\exp\left(-\frac{T}{\xi}\right)\right]^R = \exp\left(-\frac{RT}{\xi}\right) = \exp\left(-\frac{A}{\xi}\right)$

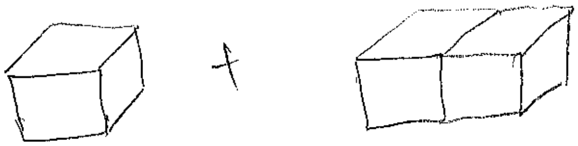
Get Area law again.

3D IGT

Recall the expansion:

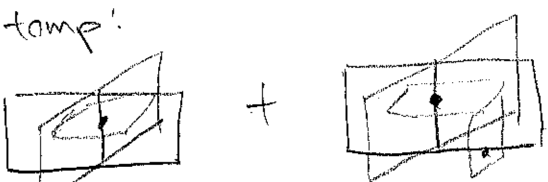
High temp:

$$Z = \sum \prod \left(1 + \overbrace{\sigma_3 \sigma_3 \sigma_3 \sigma_3 \tanh\beta}^{\text{small}}\right)$$

$= 1 +$  $+ \ldots$

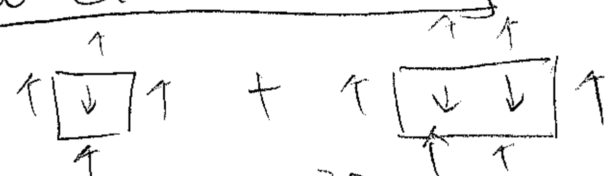
$= \sum$ closed surfaces.

Low temp:

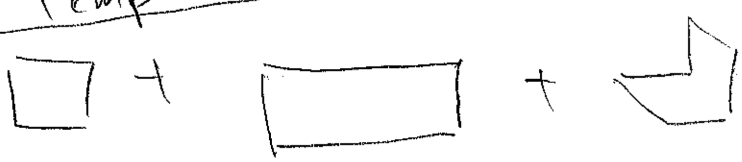
$1 +$ 

assign normal vector for frustrated plane.

Low classical Ising 3D



High Temp 3D

$1 +$ 

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3) Statistical Physics of Fields, Mehran Kardar