Low Energy Chern-Simons Effective Action for Abelian FQH Bulk and Edge

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1. Bulk
   - $\nu=\frac{1}{m}$ FQH
   - general FQH
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     - Quasi-Particle Excitation.
2. Edge
   - Edge degrees of freedom
   - Edge current
   - Edge excitation

Main Reference: Xiao-Gang Wen [cond-mat/9506066] Chpt 2&3

1. Bulk

Idea: Physical Phenomena know $\longrightarrow$ construct Eff. action reproducing the physical phenomena.

$\nu=\frac{1}{m}$ Action In FQH, the bulk phenomenon is EM current responding to EM field as:

$$-eJ^\mu = \sigma_{xy}\, \partial_\nu A_\lambda \epsilon^{\mu\nu\lambda} = \frac{\nu e^2}{2\pi}\partial_\nu A_\lambda \epsilon^{\mu\nu\lambda}$$

Here $\nu=\frac{1}{m}$, $m=1,3,5\ldots$ for fermion (electron).

To write an eff. action reproducing such response, first we write

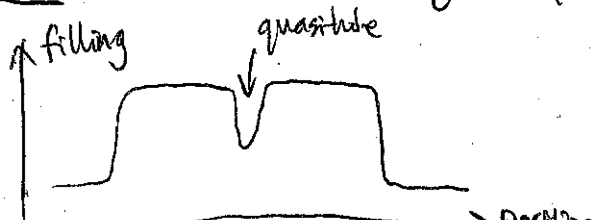
$$J^\mu = \frac{1}{2\pi}\partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} \quad (\partial_\mu J^\mu = 0 \text{ assumed}).$$

Then the eff. action

$$S = \int d^3x \left(-\frac{m}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} - \frac{e}{2\pi} A_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda}\right)$$

reproduces the σ_{xy} response above.

$\nu=\frac{1}{m}$ Quasi-Particle Now consider quasi-particle excitation in the $\nu=\frac{1}{m}$ FQH liquid.



Let the quasi-particle have an a_μ flux equal to $\frac{l}{m}$.

This means the following:

Label the quasi-particle current as J_ℓ^μ. Add the term $\int d^3x \, \ell a_\mu J_\ell^\mu$ into the action. Then $\frac{\delta S}{\delta a_0} = 0$ gives:

$$J^0 = \underbrace{-\frac{e}{2\pi m} B}_{\text{usual piece}} + \underbrace{\frac{\ell}{m} J_\ell^\mu}_{\text{new piece due to quasi-particle.}}$$

That's what we mean by the quasiparticle has a_μ flux $\frac{\ell}{m}$.

(In Laughlin wave function, this means a factor $\prod_i (z_i - \xi)^\ell$.)

If we do $\int \mathcal{D}a_\mu \, e^{iS}$, the result has terms

$$\left[\begin{array}{lll} J_\ell^\mu (\partial^{-1})^\nu J_\ell^\lambda & \longrightarrow \text{coef gives exchanging phase} & \theta_\ell = \pi \frac{\ell^2}{m}. \\ A_\mu^* J_\ell^\mu & \longrightarrow \text{coef gives eff EM charge} & Q_\ell = -e\frac{\ell}{m}. \end{array}\right.$$

One can also see the θ_ℓ by the following arguement: A quasi particle of ℓ_1 has a_μ flux $\frac{\ell_1}{m}$; another of ℓ_2 has a_μ charge ℓ_2. So winding the 2nd around the first gives a phase $2\pi \frac{\ell_1 \ell_2}{m}$. Taking $\ell_1 = \ell_2 = \ell$ and devide by 2 (since "winding phase" $= 2\theta$), we get $\theta = \pi \frac{\ell^2}{m}$. Taking $\ell_2 = m$ for electron, we see that an electron winds around a quasi-particle has phase $2\pi\ell$, which agrees with the assertion above that such quasi-particle corresponds to the factor $\prod_i (z_i - \xi)^\ell$ in wavefunction. For single-valueness, $\ell \in \mathbb{Z}$.

$\boxed{\text{General FQH Action}}$

Now consider a more generic setup, with n species of quasi-particles in condensate. These quasi-particle may be electrons (on mutiple layers), quasi-particle in electron condensate, ~~quasi-particle~~ quasi-quasi-particle in quasi-particle condensate, and so on.

$$S = \int d^3x \left(-\frac{1}{4\pi} K_{IJ} a_{I\mu} \partial_\nu a_{J\lambda} \epsilon^{\mu\nu\lambda} - \frac{e}{2\pi} A_\mu t_I \partial_\nu a_{I\lambda} \epsilon^{\mu\nu\lambda} \right)$$

$I = 1 \cdots n$, $K_{IJ} = K_{JI}$, entries of K and t are integers.

This charecterizes FQH by (K, t) up to $SL(2, \mathbb{Z})$ basis transf.

Filling fraction ν is defined s.t.:

$$-\nu e B = 2\pi t_I J_I^0.$$

The EoM is $2\pi J_I = -e K_{IJ}^{-1} t_J B$, so $\nu = t_I K_{IJ}^{-1} t_J$.

Examples:

$t = (1)$, $K = (m)$, $\nu = \frac{1}{m}$

$t = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $K = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$, $\nu = \frac{2}{5}$

$t = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $K = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$, $\nu = \frac{3}{7}$

$t = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $K = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$, $\nu = \frac{2}{5}$

$t = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $K = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, $\nu = \frac{1}{2}$

(The two $\nu = \frac{2}{5}$ are related by $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \in SL(2, \mathbb{Z})$. So they are the same; but if we also consider spin of quasi-particles, they have $s = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$ and $s = \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \end{pmatrix}$ which are NOT related by $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. So they are the same if spin is absent (or interaction mixes spins), and are different if spin is in consideration.)

$\boxed{\text{General FQH Quasi-particle}}$

Again consider a single quasi-particle with coupling $\int d^3x\, l_I a_{I\mu} J_l^\mu$.

Then it has $a_{I\mu}$ charge l_I

$a_{I\mu}$ flux $K_{IJ}^{-1} l_J$

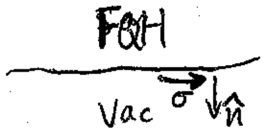
exchanging phase $\theta_l = \pi l_I K_{IJ}^{-1} l_J$

EM charge $Q_l = -e\, t_I K_{IJ}^{-1} l_J$.

2. Edge.

Edge DoF $S = -\frac{m}{4\pi}\int d^3x\, a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda}$ under $a_\mu \to a_\mu + \partial_\mu f$:

$$\Delta S = -\frac{m}{4\pi}\int_{\text{boundary}} d\sigma\, dt\, f\, \hat{n}_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda}$$



$\Delta S = 0 \iff f = \text{const. on boundary}$

$\iff \partial_t f = \partial_\sigma f = 0$ on boundary

$\iff a_t$ and a_σ do NOT transf on boundary.

To see this in another way, EoM says $\partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} = 0$

i.e. $a_\lambda = \partial_\lambda \phi$. In bulk, $\phi \to \phi + f$ is gauge inv, so ϕ in bulk are NOT dof. On edge, ϕ is dof (up to const shift).

Edge Action & Current What is the dynamics of edge dof?

Physical phenomenon: Edge current of velocity V.

(V is non-universal, magnitude is an input)

We claim that restricting on boundary

$$a_t + V a_\sigma = 0$$

will reproduce the physical phenomenon.

To see this, now, with this restriction and bulk gauge transf, we can make $a_t + Va_\sigma = 0$ everywhere. Then one find

$$S = -\frac{m}{4\pi}\int d\sigma\, dt\, (\partial_t + V\partial_\sigma)\phi \cdot \partial_\sigma \phi$$

as an edge eff. action

EoM: $(\partial_t + V\partial_\sigma)(\partial_\sigma \phi) = 0$.

If we can identify $\partial_\sigma \phi$ as current density on edge, ~~so~~ the EoM would indeed reproduce the physical phenomenon.

To see $\rho = \frac{1}{2\pi}\partial_\sigma \phi$ is edge current, consider

$$\int d^3x \ \frac{-e}{2\pi} a_\mu \partial_\nu A_\lambda \epsilon^{\mu\nu\lambda} \quad (U(1) \text{ gauge inv. is explicit})$$

Again imposing $a_t + V a_\sigma = 0$, we get

$$-\frac{e}{2\pi}\int d\sigma\, dt\, (A_t + V A_\sigma)\partial_\sigma \phi .$$

So $\rho = \frac{\partial_\sigma \phi}{2\pi}$ indeed is the edge current.

To quantize, notice that $H = \frac{mV}{4\pi}\int d\sigma (\partial_\sigma \phi)^2 = \pi m V \int d\sigma\, \rho^2$.

Since $i[H, \rho] = \partial_t \rho$, to reproduce the EoM, we need:

$$i[\rho(\sigma), \rho(\sigma')] = \frac{1}{2\pi m}\partial_\sigma \delta(\sigma - \sigma')$$

$$\Leftrightarrow \quad i[\phi(\sigma), \phi(\sigma')] = \frac{-\pi}{m}\operatorname{sign}(\sigma - \sigma')$$

(Kac-Moody Algebra) And $mV > 0$ for H to be bounded below.

More generally,

$$S = \frac{1}{4\pi}\int d\sigma\, dt \left(K_{IJ}\partial_t \phi_I \partial_\sigma \phi_J - V_{IJ}\partial_\sigma \phi_I \partial_\sigma \phi_J \right)$$

V_{IJ} non-universal, V_{IJ} positive-definite.

$$i[\rho_I(\sigma), \rho_J(\sigma')] = \frac{K^{-1}_{IJ}}{2\pi}\partial_\sigma \delta(\sigma - \sigma').$$

One can find a matrix U s.t.

$$(U K U^T)_{IJ} = \sigma_I \delta_{IJ} \quad (\sigma_I = \pm 1)$$

$$(U V U^T)_{IJ} = |V_I| \delta_{IJ} \quad (|V_I| > 0)$$

Let $V_I = \sigma_I |V_I|$, σ_I determines the direction of edge mode

σ_I is universal, $|V_I|$ is NOT.

An arguement of universality will be given later.

$\boxed{\text{Edge Excitation}}$

In bulk, $l_I a_{I\mu} J_l^\mu$ gives an excitation of $a_{I\mu}$ flux $K^{-1}_{IJ} l_J$.

On edge, we look for an operator Ψ_l that does this.

This means: $i[\rho_I(\sigma), \Psi_l(\sigma')] = l_J K^{-1}_{JI}\, \delta(\sigma-\sigma')\, \Psi_l(\sigma')$

$\Rightarrow \Psi_l \propto e^{i l_I \phi_I}$, $\Psi_l^\dagger = \Psi_l^{-1}$.

Then one can work out $\Psi_l(\sigma)\Psi_l(\sigma') = (-1)^{\lambda_l}\Psi_l(\sigma')\Psi_l(\sigma)$

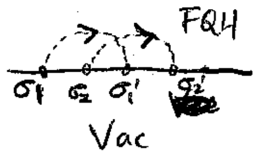
where $\lambda_l = l_I K^{-1}_{IJ} l_J$ is the edge statistics

And the excitation has charge $Q^{\Psi}_l = -e\, t_I K^{-1}_{IJ} l_J$.

Write $\tilde{l}_I \equiv U_{IJ} l_J$, $\tilde{t}_I \equiv U_{IJ} l_J$, , then:

$$\lambda_l = \sum_I \sigma_I \tilde{l}_I^2, \quad Q^{\Psi}_l = -e \sum_I \sigma_I \tilde{t}_I \tilde{l}_I$$

This provides universal argument for the Kac-Moody Alg:



bring excitation #1 from σ_1 to σ_1'
#2 from σ_2 to σ_2'

$\longrightarrow$ effectively wind #1 around #2, should get phase $2\pi\, l_I K^{-1}_{IJ} l_J$

$\Rightarrow$ Express this in $\Psi_l = e^{i l_I \phi_I}$ and working backwards, get the Kac-Moody alg.
Since the statistics should be universal, so is the alg.

One can use the commutation relations to find the propagator $\langle \Psi_l^\dagger(\sigma,t)\Psi_l(0,0)\rangle \propto (\sigma - V_I t + i\sigma_I \epsilon)^{-\tilde{l}_I^2}$

For $\sigma = 0$, $\langle \Psi_l^\dagger(0,t), \Psi_l(0,0)\rangle \propto t^{-\sum_I \tilde{l}_I^2} \equiv t^{-g_l}$

The power is NOT -1, but $-\sum_I \tilde{l}_I^2$. This can have experimental consequences (has been observed)

For system with all $\sigma_I = +1$ (-1)

$\Rightarrow g_l = \lambda_l$ $(-\lambda_l)$, universal.

Otherwise g depends on non-universal V_{IJ} via U.