Scattering Amplitudes Week 4.

Jingyuan Chen

Cachazo-Svrcek-Witten Recursion at Tree Level (Elvang & Huang Chapter 3.4)

Unitarity Method at 1-Loop Level (Elvang & Huang Chapter 6.1, 6.2)

CSW Recursion

Review of Complex shift (chapter 3.1)

External lines $p_i^\mu \xrightarrow{\text{shift}}$ complex $\hat{p}_i^\mu = p_i^\mu + z r_i^\mu$

complex r_i^μ satisfy: ① $\sum_i r_i^\mu = 0$ (so $\sum_i \hat{p}_i^\mu = \sum_i p_i^\mu = 0$)

② $r_i \cdot r_j = 0$ for any i, j

③ $p_i \cdot r_i = 0$ for each i.

Now consider a subset I of external lines.

$P_I \equiv \sum_{i \in I} p_i$, $\hat{P}_I \equiv \sum_{i \in I} \hat{p}_i$.

$$\hat{P}_I^2 = P_I^2 + z\, 2P_I \cdot \Big(\sum_{i \in I} r_i\Big) = -\frac{P_I^2}{z_I}(z - z_I) \text{ with } z_I = -\frac{P_I^2}{2P_I \cdot \big(\sum_{i \in I} r_i\big)}$$

Amplitude $\hat{A}_n$ is A_n but with $\hat{p}_i$ in place of p_i. (n is # of ext. lines)

Cauchy Thm used on $\frac{\hat{A}_n(z)}{z} \Rightarrow A_n = -\sum_{z_I} \operatorname{Res}_{z = z_I} \frac{\hat{A}_n(z)}{z} + \hat{A}_n(\infty)$

(This relies on that at tree level the only analytic structures are poles)

$$\operatorname{Res}_{z = z_I} \frac{\hat{A}_n(z)}{z} = \operatorname{Res}_{z = z_I} \frac{1}{z}\Big(\ \Big)$$

$$= \operatorname{Res}_{z = z_I} \frac{\hat{A}_L(z)\, \hat{A}_R(z)}{z \left(-\frac{P_I^2}{z_I}\right)(z - z_I)}$$

$$\Rightarrow A_n = \sum_I \frac{\hat{A}_L(z_I)\hat{A}_R(z_I)}{P_I^2} + \hat{A}_n(\infty)$$

$\hat{A}_n(\infty)$ depends on the theory and the shifts r_i.

Usually ~~it~~ is preferable to choose the shifts s.t. $\hat{A}_n(\infty) = 0$.

In BCFW, the choice of shift is s.t., choose i_1, i_2, and

$$\begin{cases} r_{i_1}^{\mu} = -r_{i_2}^{\mu} = -\frac{1}{2}\langle i_1|\gamma^{\mu}|i_2] \\ r_i^{\mu} = 0 \quad \text{for } i \neq i_1, i \neq i_2 . \end{cases} \iff \begin{array}{ll} |\hat{i}_1] = |i_1] + z|i_2], & |\hat{i}_1\rangle = |i_1\rangle, \\ |\hat{i}_2] = |i_2], & |\hat{i}_2\rangle = |i_2\rangle - z|i_1\rangle . \end{array}$$

Advantage: few non-vanishing terms in recursion
Drawback: involves shift in $|\,]$ as well as $|\,\rangle$.

CSW: For all i,

$$r_i^{\mu} = -\frac{c_i}{2}\langle i|\gamma^{\mu}|X] \iff |\hat{i}] = |i] + z c_i |X], \quad |\hat{i}\rangle = |i\rangle$$

for some choice of $|X]$ (chosen for convenience) and $\sum_{i=1}^{n} c_i |i\rangle = 0$

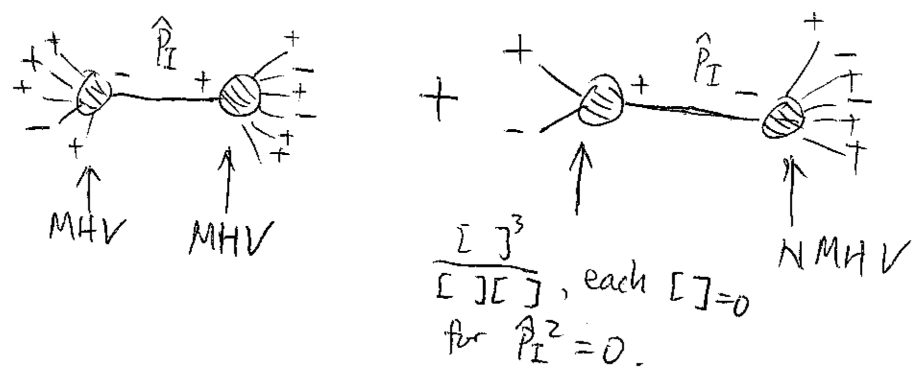
Advantage: Only involves shift in $|\,]$
Drawback: more non-vanishing terms in recursion.

Exploit the Advantage:

Construct amplitudes from MHV amplitudes, where Parke-Taylor says: $A_n(1^+ \cdots i_1^- \cdots i_2^- \cdots n^+) = \dfrac{\langle i_1 i_2\rangle^4}{\langle 12\rangle \cdots\cdots \langle n1\rangle}$ which involves no $|\,]$.

For example, consider NMHV, with three "$-$" and the rest "$+$".

$A_n^{NMHV} =$



$$= \frac{A_L^{MHV}(\hat{P}_I)\, A_R^{MHV}(\hat{P}_I)}{P_I^2} .$$

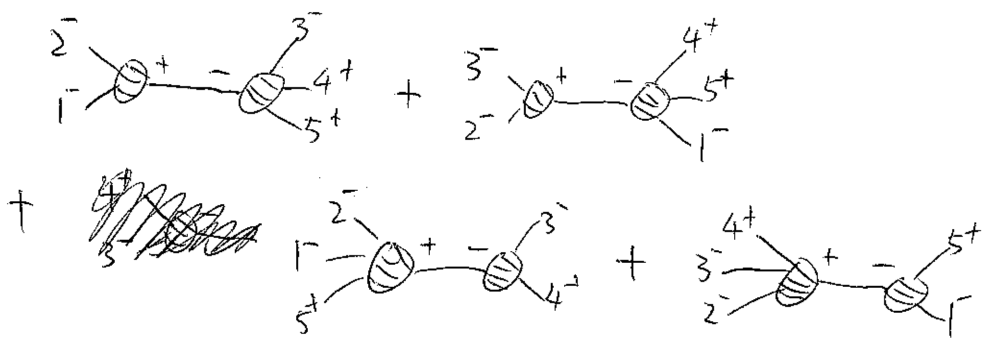
The only place the shift $|X]$ shows up is in $|\hat{P}_I\rangle$ in A_L^{MHV} and A_R^{MHV}.

$$|\hat{P}_I\rangle = |\hat{P}_I\rangle \frac{[\hat{P}_I X]}{[\hat{P}_I X]} = \frac{\hat{\not{P}}_I |X]}{[\hat{P}_I X]} = \frac{\not{P}_I |X]}{[\hat{P}_I X]}$$

Moreover, by little group scaling, the $[\hat{P}_I X]$ factors always cancel out, So effectively $|\hat{P}_I\rangle \to \not{P}_I |X]$.

e.g. $A_5(1^- 2^- 3^- 4^+ 5^+) = \dfrac{[45]^4}{[12][23][34][45][51]}$ from Parke-Taylor.

Can also compute it by



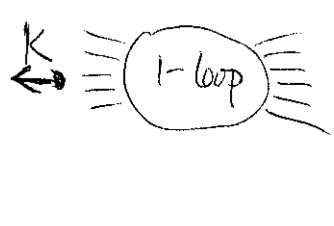
with $|X] = |2]$

Unitarity Method.

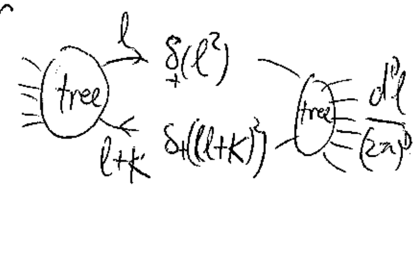
Complication: At loop level, internal lines in the loop can go on-shell simultaneous, leading to branch cuts in complex momentum space.

Can use this, and unitarity, to reconstruct 1-loop amplitude.

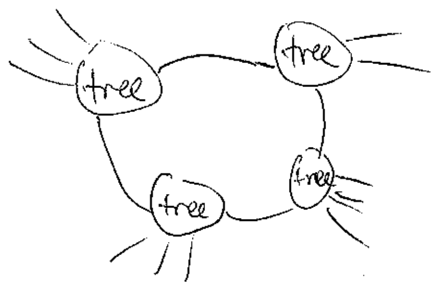
$$SS^\dagger = 1, \quad S = 1 + iT \implies -i(T - T^\dagger) = T^\dagger T$$



has discontinuity in K given by $\int$



In the loop, how many propagators can go on-shell simultaneously?

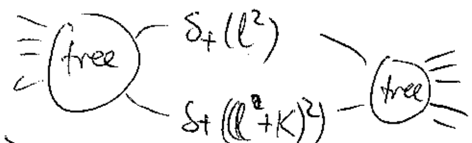


At most 4, because
$$\ell^2 = (\ell-K_1)^2 = (\ell-K_1-K_2)^2 = \cdots (\ell-K_1-\cdots-K_{n-1})^2 = 0$$
has n equations, ℓ has 4 components so generally $n \leq 4$.

One can parametrize
$$A_n^{\text{1-loop}} = \sum_{\alpha_4} C_4^{\alpha_4} I_4^{\alpha_4} + \sum_{\alpha_3} C_3^{\alpha_3} I_3^{\alpha_3} + \sum_{\alpha_2} C_2^{\alpha_2} I_2^{\alpha_2} + (\text{rational})$$

α_2 labels ways of separating ext. lines into two sets,
$$I_2^{\alpha_2} \equiv \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 (\ell-K_1^{\alpha_2})^2}.$$
Likewise for I_3 and I_4.

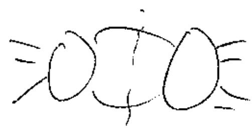
Identify the 4-pole ~~singularity~~ singularities from $C_4^{\alpha_4} I_4^{\alpha_4}$ with ~~those~~ those from



with two ~~more~~ extra poles in ℓ. $\Rightarrow$ Get $C_4^{\alpha_4}$
$$C_4^{\alpha_4} = \frac{1}{2} \sum_{\substack{\ell=\ell_*^1,\\ \ell=\ell_*^2}} A_1^{\text{tree}}(\ell)\, A_2^{\text{tree}}(\ell)\, A_3^{\text{tree}}(\ell)\, A_4^{\text{tree}}(\ell)$$

$\ell_*^{1,2}$ are the ~~good~~ solution to the quadratic constraint of 4 propagators on-shell

Similarly,



has terms with one extra pole, get $C_3^{\alpha_3}$
no extra pole, get $C_2^{\alpha_2}$.

Notice that only I_2 is UV div. Moreover, all the $I_2^{\alpha_2}$'s have the same leading $\frac{1}{\epsilon}$ from dim-reg. So for any dimensionless parameter, $\beta \propto \sum_{\alpha_2} C_2^{\alpha_2}$.

The rational terms arise from the extra dimension
$2\epsilon = 4-d$ from dim. reg.

They are the most difficult to compute. The way is,
the unitarity cut really is performed in $d=4-2\epsilon$ dim,

$$l = l^{(4)} + \mu^{2\epsilon}. \quad l^2=0 \Rightarrow l^{(4)} = (-\mu^2) \leftarrow \text{a mass term.}$$

Extract from $I_m^{\alpha_m\,(d)} = \int \frac{d^4 l^{(4)}}{(2\pi)^4} \frac{d^{2\epsilon}\mu}{(2\pi)^{2\epsilon}}$ (usual, with mass μ^2).

For SUSY theories rational terms vanish.

For $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA, only $C_4^{\alpha_4}$ non-vanishing.