(3+1)D Weyl fermion and IQHE in (2+1)D

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I. Weyl fermion as Berry curvature monopole

II. Effective Weyl fermion in band model; fermion doubling

III. Connection to IQHE

① Chern number

② edge mode

~~③ Landau level~~

Ref: Hasan & Kane, II.B.2,3
Haldane, PRL 61, 2015 (1988)

I. Weyl Eq: $H\psi = \overset{RH}{\underset{LH}{\pm}} \sigma^i p^i \psi \Longrightarrow \pm|\vec{p}|\psi = \sigma^i p^i \psi$

K.E. +: RH, $h=+\frac{1}{2}$ OR LH, $h=-\frac{1}{2}$
K.E. −: RH, $h=-\frac{1}{2}$ OR RH, $h=+\frac{1}{2}$

solve for ψ: For $\vec{p}=|\vec{p}|\hat{z}$, have $\psi_+ = \begin{pmatrix}1\\0\end{pmatrix}$ $(h=+\frac{1}{2})$ and $\psi_- = \begin{pmatrix}0\\1\end{pmatrix}$ $(h=-\frac{1}{2})$

General $\vec{p} = (|\vec{p}|, \theta, \varphi)_{sph.}$, apply Euler angles $e^{-i\varphi\frac{\sigma^3}{2}} e^{-i\theta\frac{\sigma^2}{2}} e^{-i\alpha\frac{\sigma^3}{2}}$ to $\psi_\pm$.

$$\Rightarrow \psi_+(\hat{p}) = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-\frac{i}{2}(\alpha+\varphi)} \\ \sin\frac{\theta}{2}\, e^{-\frac{i}{2}(\alpha-\varphi)} \end{pmatrix}, \quad \psi_-(\hat{p}) = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{\frac{i}{2}(\alpha-\varphi)} \\ \cos\frac{\theta}{2}\, e^{\frac{i}{2}(\alpha+\varphi)} \end{pmatrix}$$

α: gauge freedom of Berry connection, due to overall phase of ψ.

Berry curvature: $\Omega_{\pm\, ij} = -i\left(\partial_i \psi_\pm^\dagger \partial_j \psi_\pm - (i\leftrightarrow j)\right) = \pm\frac{1}{2}\frac{\epsilon_{ijk}\hat{p}_k}{|\vec{p}|^2}$

↳ monopole of charge 2π at $\vec{p}=0$.

⇝ This suggests a better understanding of Ω.

Each $\hat{p}$ maps to a $U(1)$ fibre inside $SU(2)\simeq S^3$. So we are considering $U(1)$ bundle on $S^2 \ni \hat{p}$. So charge of Ω can only come from $\vec{p}=0$ which charecterizes the non-trivial topology of S^2 (for massless particle). More over, problem is sph. symm. So $\Omega_{ij} \propto \frac{\epsilon_{ijk}\hat{p}_k}{|\vec{p}|^2}$.

Moreover, $C_1 = \frac{1}{2\pi} 4\pi |\vec{p}|^2 |\Omega| \Rightarrow \Omega_{ij} = \frac{C_1}{2}\frac{\epsilon_{ijk}\hat{p}_k}{|\vec{p}|^2}$.

This C_1 is for $U(1)$ bundle on S^2 of $\hat{p}$. Looking back at α, we see the $U(1)$ is also the little group of $\hat{p}$. So by def, $C_1 = 2h$. in this case $C_1 = \pm 1$.

Summary: $C_1 = 2h = +1$: RH, K.E. >0 OR LH, K.E. <0
$C_1 = 2h = -1$: RH, K.E. <0 OR LH, K.E. >0.

II. Consider 2-band $H\psi = \left(K^i(k^j)\,\sigma^i + K^0(k^j)\right)\psi$.

If for some k^j, all $K^i = 0$, then bands touch.

k^j: 3 dof , $K^i = 0$: 3 eqs $\longrightarrow$ band touching is robust against perturbations.

Near touching point $\vec{k}_{deg}$, expand to 1st order:

$$\cancel{H - K^0(k^j_{deg}) + (k^j - k^j_{deg})(}$$

$$\underbrace{H - K^0(k^j_{deg}) - (k^j - k^j_{deg}) \left.\frac{\partial K^0}{\partial k^j}\right|_{\vec{k}_{deg}}}_{\to\ \omega} = \sigma^i \underbrace{(k^j - k^j_{deg}) \left.\frac{\partial K^i}{\partial k^j}\right|_{\vec{k}_{deg}}}_{\to\ p^i}$$

Have $\omega\psi = \sigma^i p^i \psi$. $\mathrm{Det}\left(\frac{\partial K^i}{\partial k^j}\right) \begin{array}{l} >0 \longrightarrow \text{RH in terms of } \vec{k} \\ <0 \longrightarrow \text{LH in terms of } \vec{k}. \end{array}$

$\mathcal{T}: \vec{k} \to -\vec{k},\ \sigma^i \to \sigma^{i*} = -\sigma^2\sigma^i\sigma^2$, $\quad \mathcal{P}: \vec{k} \to -\vec{k},\ \sigma^i \to \sigma^1\sigma^i\sigma^1$, $\quad \mathcal{C}: \vec{k} \to \vec{k},\ \sigma^i \to -\sigma^3\sigma^i\sigma^3$

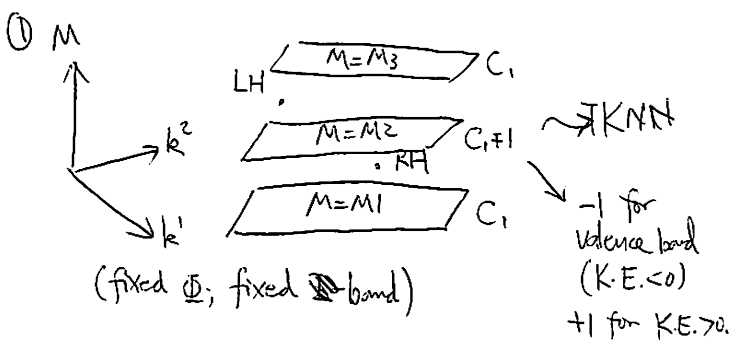
Comment: This is NOT the well-known "doubling theorem". The "doubling theorem" requires a lattice and is NOT subjected to perturbation from symmetry. See: Nielsen & Ninomiya, Nucl. Phys B 185, 20 (1981).

H is $\begin{cases} \mathcal{T} \text{ inv} \Rightarrow K^0, K^1, K^3 \text{ even in } \vec{k},\ K^2 \text{ odd in } \vec{k} \\ \mathcal{P} \text{ inv} \Rightarrow K^0, K^1 \text{ even in } \vec{k},\ K^2, K^3 \text{ odd in } \vec{k} \end{cases}$

If both inv, $K^3 = 0$, no Weyl fermion. If only one is inv., then an RH/LH Weyl fermion at $\vec{k}$ implies an LH/RH Weyl fermion at $-\vec{k}$.
$\rightsquigarrow$ fermion doubling due to $\mathcal{P}, \mathcal{T}$.
Now add perturbation so that $\mathcal{T}$ not inv, $\mathcal{P}$ not inv., but Weyl fermions are robust, so still have the fermion doubling.

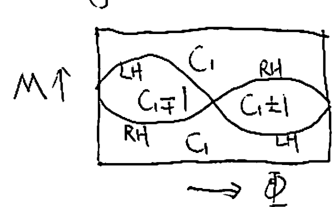
III. Now consider 2D, $H = H(k^1, k^2; M; \Phi)$ where M is some system parameter that we take to play the role of k^3; other parameters are collectively called Φ.

①

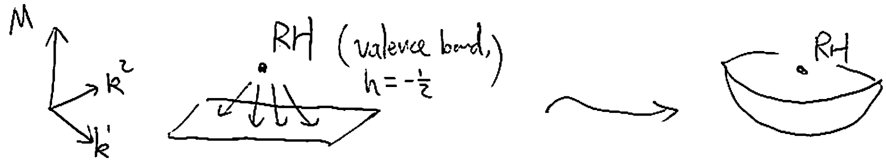


(fixed Φ; fixed band)

Φ can adjust positions of the Weyl fermions, so get phase diagram like:



The Berry curvature flux through a 2D BZ due to one Weyl fermion:

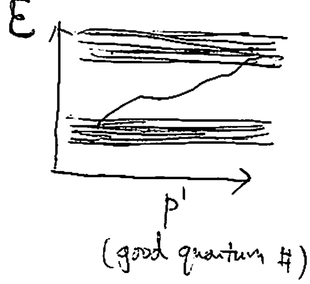


$\frac{1}{2\pi}\int \Omega = -\frac{1}{2}$

Doubling of fermion ensures the total flux is $\sum_{i^{th}\text{ pair}} \left(\pm\frac{1}{2} \pm \frac{1}{2}\right) \in \mathbb{Z}$.

i.e. $C_1 \in \mathbb{Z}$.

② To relate these discussion to bulk/edge, think of M changing macroscopically as $M(x_2)$ in the x_2 direction of sample.
Say $M(x_2)$ changes so that the BZ pass through a monopole.
(in fact Φ can change as well)
Solve for the Weyl Eq. $H\psi = \left(\sigma^{1,2}(-i\partial_{1,2}) + m(x_2)\sigma^3\right)\psi$,
get edge state



which is chiral.

Extra: The chiral state above $\rightsquigarrow$ (1+1)D RH Weyl fermion.
But on this edge, NO corresponding LH.
$\rightsquigarrow$ edge state of (2+1)D system violates (1+1)D Weyl fermion doubling theorem (the rigorous one, in lattice, proven in Nielsen & Ninomiya, Nucl. phys B 185, 20 (1981)).
Similarly, boundary state of (4+1)D system violates (3+1)D Weyl fermion doubling theorem $\rightsquigarrow$ "Domain Wall".