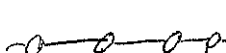
Basics of MERA | Intro

Last week: we saw tensor networks like MPS, PEPS.
These mimic the geometry of the system being modeled,

e.g.  for spin chain

and work well for gapped systems: $S_A \leq 2 \log \chi$ (χ = size of bonds, basically ent. independent of size.)

For gapless systems, a better alternative is the Multiscale Entanglement Renormalization Ansatz (MERA).

Basic idea is to add another dimension to the network, corresponding to RG flow. This should sound a bit like the radial direction in AdS/CFT, as we will make more precise later.

Basics of MERA

We want to implement RG in real-space, corresponding to a coarse-graining transformation. In more detail, take a 1D lattice system $\mathcal{L}$ → Hilbert space $\mathcal{H} = \bigotimes V_i$

implement coarse-graining with an isometry,

$$w: V_{s'} \longrightarrow V_{s_1} \otimes V_{s_2}, \qquad w^\dagger w = \mathbb{1}_{V_{s'}}, \quad w w^\dagger = P$$

states mapped $w^\dagger |\psi\rangle = |\psi'\rangle \in V_{s'}$.

operators, e.g. Hamiltonians mapped as

$$H' = w^\dagger H w.$$

How faithful is this transformation? Look at the density matrix

$$\rho = \mathrm{Tr}_{\text{env}} |\psi\rangle\langle\psi| = \sum_{\alpha=1}^{\chi} p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|, \qquad \sum_\alpha p_\alpha = 1.$$

Then we should choose w s.t.

$$P = w w^\dagger = \sum_{\alpha=1}^{\chi} |\psi_\alpha\rangle\langle\psi_\alpha|$$

In practice, we can neglect α with smallest p_α's, accepting

an error ε s.t.

$$1 - \sum_{\alpha=1}^{\chi} p_\alpha \leq \varepsilon \ll 1.$$

χ is a measure of entanglement between r_1 / r_2 and the rest of the system:

$$S = -\sum_{\alpha=1}^{\chi} p_\alpha \log p_\alpha \leq \log \chi \,, \quad \text{saturated for } p_\alpha = 1/\chi.$$

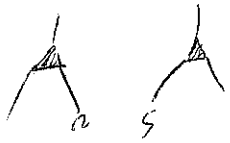
For critical systems, if we repeat this many times, we need larger and larger χ for accuracy. This reflects a failure of the simple coarse-graining; it is affected by entanglement at scales smaller than the graining we have, if the regions are at the boundary.

For example: Take two sites r, s, in a state $\frac{|\uparrow\rangle_r|\uparrow\rangle_s + |\downarrow\rangle_r|\downarrow\rangle_s}{\sqrt{2}}$.

If coarse-graining does 

, don't contribute to entanglement between $r \oplus s$ and rest $\Rightarrow$ removed, i.e., s' is in a pure state.

If instead they belong to different blocks,



entanglement never removed!

So this retains short-range features: not a proper RG flow.

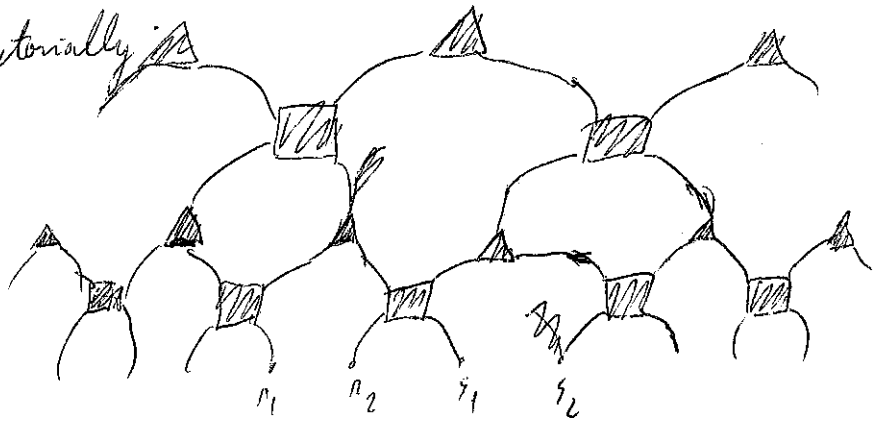
(Still works for 1D gapped systems, DMRG)

We can modify the procedure by adding disentanglers,

$u : V^{\otimes 2} \to V^{\otimes 2}$, $\quad u u^\dagger = u^\dagger u = \mathbb{1}$,

along the boundary of 2 blocks before we coarse-grain them, such that u removes the entanglement between the boundary.

Pictorially



Again, map $H' = w^\dagger u^\dagger H u w^\dagger$

The density matrix now has a smaller rank, that in particular does not increase, even for 1D critical systems, as the procedure goes on.

Note we can map operators up the layers of the network,

$O \to O' = w^\dagger u^\dagger O u w$ goes up,

and density

Our final tensor network MERA is the collection of all disentanglers and isometries. For a translation-inv. system, all u, w same for each layer, depending on $O(\chi^4)$ parameters. Since there are $O(\log N)$ layers, whole thing gives

$$O(\chi^4 \log N).$$

Also true for 2D systems in the form $O(\chi^9 \log N)$.

In practice, this is what is optimized to find $|\Psi_{GS}\rangle$ of a given Hamiltonian.

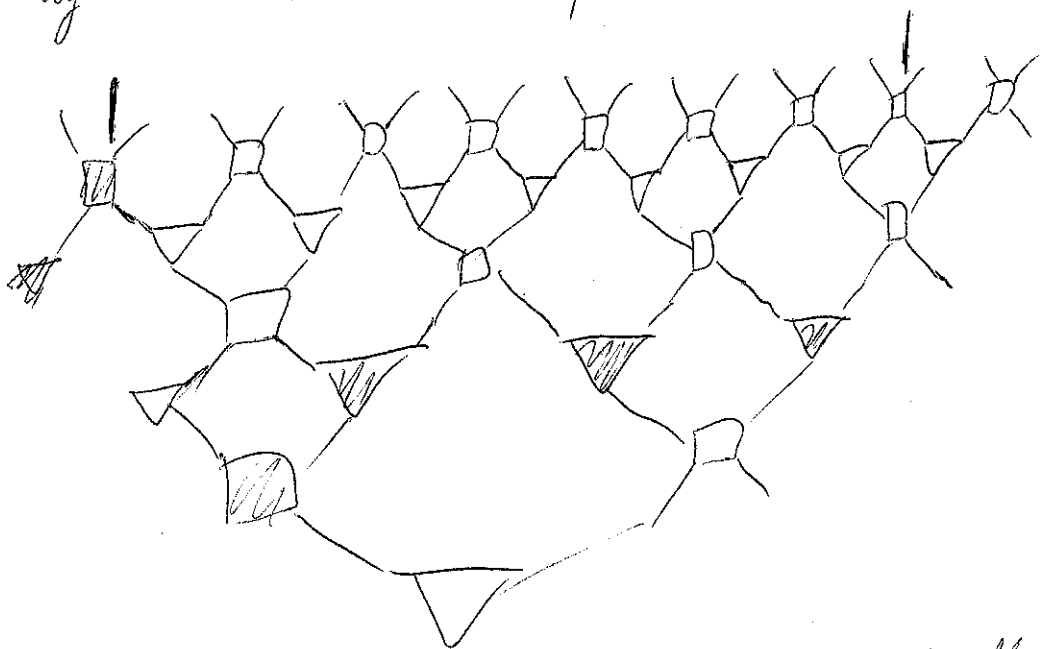
On a critical system, since it is scale-invariant, the picture is improved: disentanglers and isometries should have the same form for all layers, only two tensors u, w to optimize on.

This method was used with $\chi = 16$ to find scaling dimensions of operators of 1D Ising chain. Accurate up to 0,02%.

- One advantage of MERA is it makes manning of scale invariance picture on a lattice, without taking continuum limit.
- MERA can be seen as a quantum circuit. We will see more of this viewpoint next week.
- Support of an operator does not grow.

Behavior of entanglement entropy:

to compute S_A for a block of lattice sites, need the density matrix. How to compute it?



We get entropy from tracing out that occurs when one cut a bond. So we get an upper limit for the entanglement entropy, interpretible as a length / area in a $(d+1)$-dimensional space (round furnion?)

$S_A \leq n \log \chi$ ← bonds cut

Can we express this length as coming from a metric?

~~For a critical system, each layer is identical, and we get [illegible]~~
~~[illegible] we break~~

How many bonds do we break at each layer?

1D: 2.

Layers needed: $\log_2(L)$

$S_A \le 2 \log_2 L \log \chi$

dD: $\left(\frac{L}{2^z}\right)^{d-1}$

Layers: $\log_2(L)$

$$S_A \lesssim L^{D-1} \sum_{z=0}^{\log_2(L)} (2^{-z})^{d-1} \log\chi \simeq \partial A \log \chi$$

Interpret this as area: for z labelling layer,

$y = e^z$

$dy = dz\, e^z = y\, dz$

$$ds^2 = R^2 \left(dz^2 + e^{-2z} dx^i dx^i\right)$$
$$= R^2 \left(\frac{dy^2}{y^2} + \frac{dx^i dx^i}{y^2} \right)$$

$\rightsquigarrow$ spatial part of AdS metric! (state at fixed time)

(for the $\frac{dx^i dx^i}{y^2}$ part:) for this part, count bonds you would cut if you closed surface on that layer.

Note original system is at small z, small y ... Boundary of AdS.

~~[illegible]~~

RT backwards: start from entanglement, build geometry from structure of entanglement in the tensor network.

Notes

- If correlation length is finite, entanglement is removed after finite number of steps. Space ends.
- If we start with a thermal state, at some point in the process we find a density matrix for a fully mixed state, proportional to $\mathbb{1}$. This looks like ∞ temperature $\Rightarrow$ horizon of a BH?
- Where do things like large N enter? For $S \propto \partial A$, need bonds to be maximally entangled. If not, this changes, and we need the geometric interpretation to change ...
  - Use this to move away from large N, AdS?
  - How do we use this to find time-component?
  - Quantum circuit...

Refs

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1106.1082 - Evenbly, Vidal } MERA & geometry
0905.1317 - Brian Swingle