Group Cohomology and SPT phases classification

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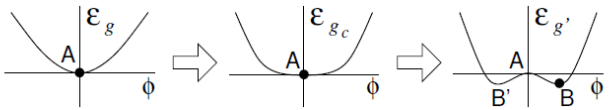
Outline

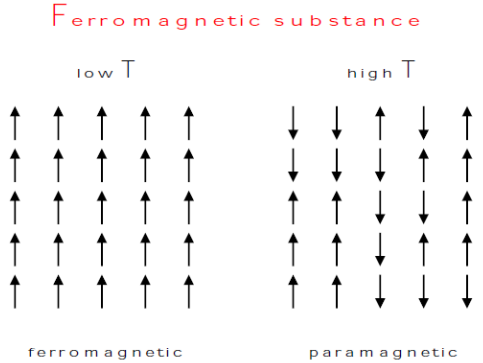
Phases of matter

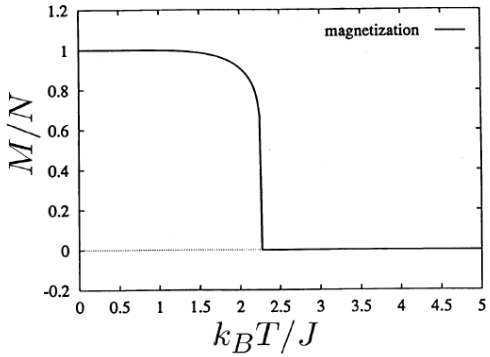
Landau's theory

Due to the idea of Landau, to have phase transition, we need order parameter and symmetry breaking $G \to H$. Different phases correspond to different symmetries.
Example 2D Ising model of Ferromagnet

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} s_i s_j - h \sum_i s_i, \quad s_i = \pm 1, \quad i = 1, .., N \tag{1}$$





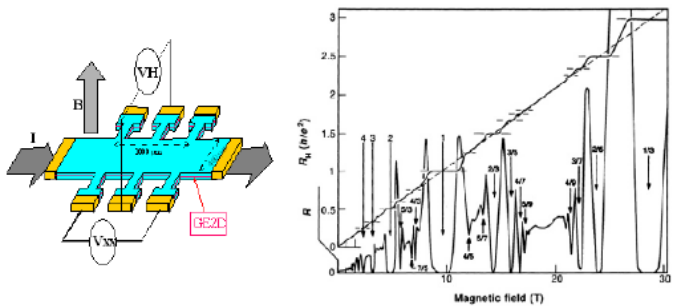


The rotational symmetry is broken, the magnetization density $\langle s_i \rangle = M/N$ plays the role of (local) order parameter.

Phases of matter

Topological order

There are phases of matter that can not be described by order parameters (Landau's theory) or symmetry breaking. For example Fractional Quantum Hall system

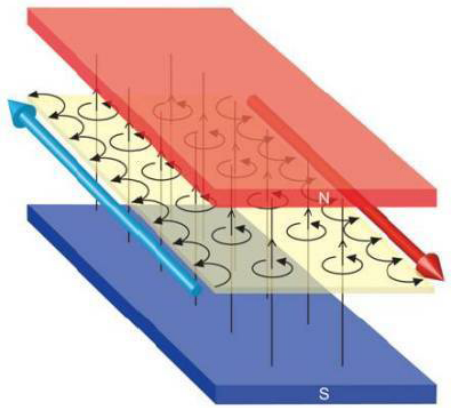


There is no symmetry breaking or the conventional order parameter, so FQH should have new order, which is topological order.

FQH liquid has internal pattern. In Laughlin state

$$\Psi_{Laughlin} = \prod_{i<j}(z_i - z_j)^m e^{-\sum_i |z_i|^2/2l_B^2}, \tag{2}$$

1. Electrons moves around clock-wise (only $z = x + iy$ dependent) in circles which are quantized Landau's orbits .
2. Electron goes around one other and gains phase $2m\pi$ each round, m is odd due to Pauli principle.
3. Each electron does not only move around 1 electron but moves around all other electrons to make a global pattern.



Phases of matter

Topological order

The global pattern shows the long range entanglement property of FQH. It is also the topological order in FQH state.
The topological order is partly characterized by the degeneracy of ground state. In the case of Laughlin state on Riemann surface, the degeneracy of ground state is m^g, in which g is the genus of Riemann surface.
The degeneracy is topological, and it does not due to the symmetry of Hamiltonian, since it does not depend on perturbations such as impurities.
We have seen that the topological phases have two properties

1. Long range entanglement
2. No symmetry breaking

Are there any non-trivial phases with short range entanglement and without symmetry breaking?

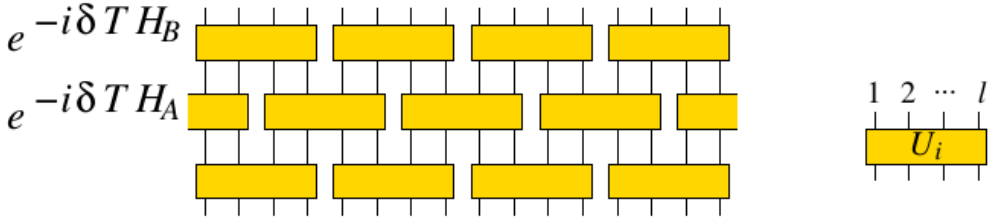
Short range entanglement and SPT phases

Local Unitary transformation

We consider the gapped system (we will only talk about ground states), Two states $|\Psi(1)\rangle$ and $|\Psi(0)\rangle$ are connected by local unitary evolution if

$$|\Psi(1)\rangle = P\left(e^{i\int_0^T dt\tilde{H}(t)}\right)|\Psi(0)\rangle \qquad P \text{ is the path ordering operator} \quad (3)$$

where we used a fake Hamiltonian $\tilde{H}(t) = \sum_i O_i(t)$ and $O_i(t)$ are local Hermitian operators



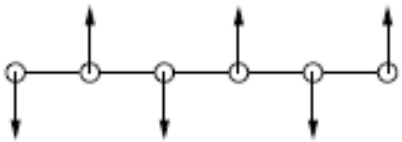
Any LU transformation can be described by a finite-depth quantum circuit LU transformation

$$|\Psi(1)\rangle = \prod_t \left(e^{i\delta t\sum_i O_i(t)}\right)|\Psi(0)\rangle \qquad \left(e^{i\delta t O_i(t)} = U_i(t)\right) \qquad (4)$$

Short range entanglement and SPT phases

Local Unitary transformation

The short range entangled states are defined as the states that are connected to a product state by local unitary (LU) evolution.

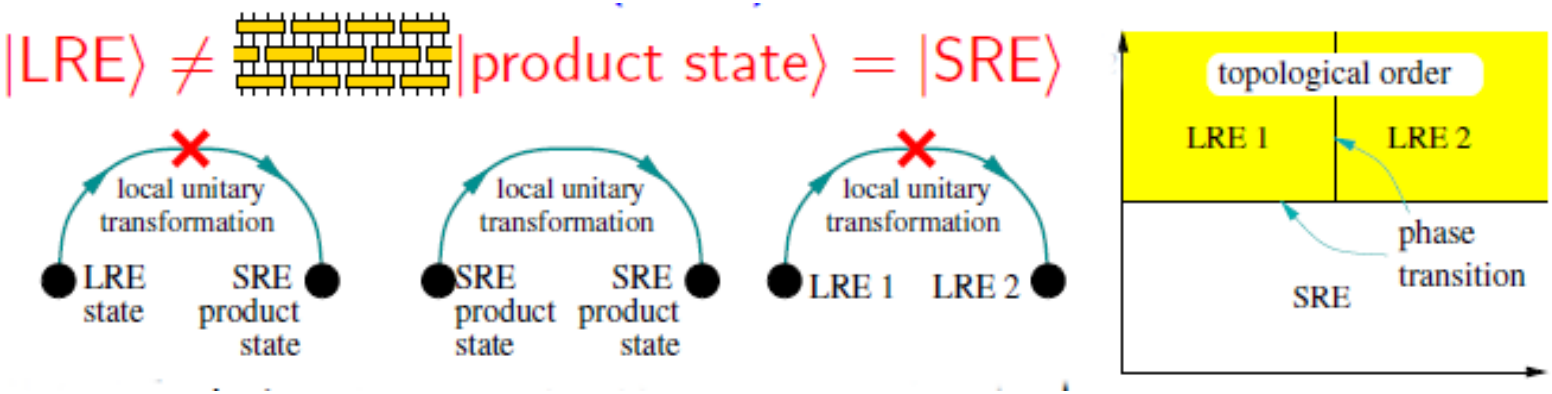
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \ldots \rightarrow$ unentangled
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes \ldots \rightarrow$ short-range entangled (SRE) state

By the definition, without any symmetry to protect (there is no restriction on LU transformations), all SRE states belong to single trivial phase since all of them can be derived from each other by a LU transformation.

Short range entanglement and SPT phases

Short range entangled states

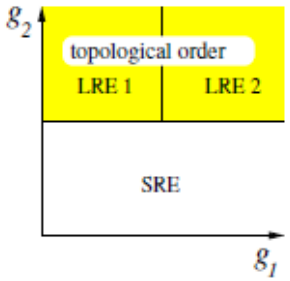
We consider the gapped system, without symmetry breaking, we can distinguish the phases of long range entangled states based on topological oders. However we may think of short range entangled state as a single trivial phase.

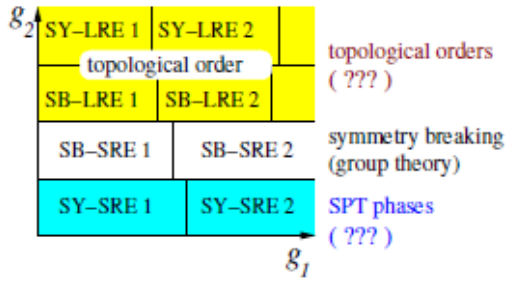


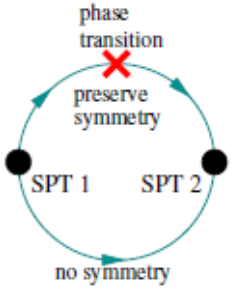
Short range entanglement and SPT phases

SPT phases

For gapped systems with a symmetry $H = U_g H U_g^\dagger$, in which $g \in G$ is a element of the symmetry group.
If we require the LU transformations (can be considered as time evolution operator of a specific Hamiltonian) must preserve the symmetry, not all SRE states can be connected to each other by a symmetry preserving LU transformation.
SPT phases=equivalent classes of symmetric LU transformations, or we may thing of SPT phases as the ground states of equivalent classes of symmetric Hamiltonians.







How to classify all SPT phase in $d+1$ dimensions with symmetry group G (here we only consider G is internal symmetry (on-site symmetry))?

Non linear sigma model (NLσM)

A SPT phase can be described by a non-linear σ model of a field $\mathbf{n}(x,\tau)$, in imaginary-time, the partition function is

$$Z = \int D\mathbf{n} e^{-\int d^d x d\tau \mathcal{L}[\mathbf{n}(\mathbf{x},\tau)]} \tag{5}$$

In order to consider symmetry $\mathbf{n} \to g.\mathbf{n}$, the action-amplitude has to be invariant under the (global) transformation

$$e^{-\int d^d x d\tau \mathcal{L}[\mathbf{n}(\mathbf{x},\tau)]} = e^{-\int d^d x d\tau \mathcal{L}[g.\mathbf{n}(\mathbf{x},\tau)]} \tag{6}$$

To study SPT phase, we can extend to the NLσM whose the field takes value in the symmetry group G

$$Z = \int D\mathbf{g} e^{-\int d^d x d\tau \mathcal{L}[g(x,\tau)]} \tag{7}$$

We consider d+1 D system with the action

$$S = \int d^d x d\tau \frac{1}{2\lambda} Tr\left(|g^{-1}\partial g|^2\right) \tag{8}$$

The action is invariant under the global transformation $g(x) \to hg(x)$, where h and g are element in G.
But the ground state is invariant under the symmetry or not depend on the parameter λ

- If λ is small the ground state is ordered $\langle g(x,\tau)\rangle = g_0$, the symmetry is broken.
- If $\lambda \to \infty$, the ground state is disordered $\langle g(x,\tau)\rangle = 0$ (due to quantum fluctuations), the symmetry is preserved.

We want to consider the symmetry preserving ground state, so under RG flow, $\lambda \to \infty$, the disordered symmetric phase is described by a fixed point theory $S_{fixed} = 0$ or $e^{-S_{fixed}} = 1$, the amplitude for any path in the path integral is 1.

We may think that the disordered ground states are trivial, since no excitation at low energy.
But the disordered ground states can be non-trivial, they can belong to different phases, even without symmetry breaking.
We would like to consider NLσM that describe SRE phase with finite energy gap and finite correlations length. So the low energy fixed point action amplitude

$$e^{-\int d^d x d\tau \mathcal{L}_{fixed}[g(x,\tau)]} \tag{9}$$

must not depend on the space time metrics (if we ignore gravitational anomaly). In other words, the fixed point NLσM must be a topological quantum field theory or topological NLσM .
A trivial topological NLσM is given by fixed point Lagrangian $\mathcal{L}_{fixed}[g(x,\tau)] = 0$ which describe the trivial SPT phase.

Since the dynamical term of NLσM vanish at fixed point in disordered phase, we can add different topological terms to the action. Different disordered phases correspond to different topological terms
We can add topological terms to the original NLσM

$$S_{fixed} = \int d^d x d\tau \left(\frac{1}{2\lambda} Tr \left(|g^{-1} \partial g|^2 \right) + 2\pi i W_{G-top} \right) \tag{10}$$

where $W_{G-top}[g(x,\tau)]$ is a topological term, here we consider quantized θ term. This term can be appeared if $\pi_{d+1}(G) \neq 0$.
Under RG flow to low energy, the dynamical term vanishes, the quantization of θ term make sure that the action amplitude is trivial for the disordered phase $e^{-S_{fixed}} = 1$.
Further more, the quantized θ term is (global) symmetry invariant.

Some example of quantized θ term

Consider the space-time manifold is closed and compact $M = S^{d+1}$. The field configuration $g(x,\tau)$ is the map $g : S^{d+1} \to G$ or an element of $\pi_{d+1}(G)$

- 0+1 D with $G = U(1) = S^1 (g = e^{i\theta})$

$$W_{G-top}[g(x,\tau)] = -k\frac{1}{2\pi}g^{-1}\partial_\tau g = k\frac{\dot{\theta}}{2\pi}, \qquad k \in Z \tag{11}$$

$$\int d\tau W_{G-top}[g(x,\tau)] = k \times \text{winding number} \tag{12}$$

- 2+1 D with $G = SU(2) = S^3$

$$W_{G-top}[g(x,\tau)] = k\frac{1}{24\pi^2}\epsilon^{\mu\nu\lambda}Tr\left[(ig^{-1}\partial_\mu g)(ig^{-1}\partial_\nu g)(ig^{-1}\partial_\lambda g)\right], \qquad k \in Z \ (1$$

$$\int d\tau W_{G-top}[g(x,\tau)] = k \times \text{winding number} \tag{14}$$

The topological term has no dynamical effect $e^{-S_{fixed}} = 1$, but can give rise to different symmetric phases classified by k.

Non linear sigma model (NLσM)

Classification of topological terms

The disordered symmetric ground state is described by fixed-point theory

$$S_{fixed} = 2\pi i \int W_{G-top}[g(x,\tau)] \tag{15}$$

The fixed point theory (2π quantized topological terms) correspond to symmetric phases
The symmetric phases are classified by the quantized topological term

$$Hom(\pi_{d+1}(G), Z) \tag{16}$$

which is represented by k.
However, in the $\lambda \to \infty$ limit, $g(x,\tau)$ is not a continuous function. The mapping classes $\pi_{d+1}(G)$ does not make sense. We need to discretize the NLσM or we need the lattice version of topological NLσM.

Lattice topological NLσM and group cohomology

Definition of Lattice topological NLσM by path integral

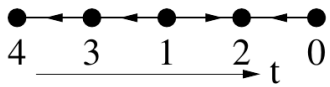
- 0+1 D space-time lattice path integral

$$Z = \sum_{\{g_I\}} \prod_I \nu_1(g_{I+1}, g_I), \qquad \nu_1(g_i, g_j) = (e^{-\int_{\_} \mathcal{L}_{fixed}})_{g_i, g_j} \tag{17}$$

  where I is time ordered index.

- With this specific triangularization of (space)-time, we have

$$\begin{aligned} Z &= \sum_{g_0,\dots,g_4=g_0} \nu_1(g_0,g_2)\nu_1(g_2,g_1)\nu_1(g_1,g_3)\nu_1(g_3,g_4) \\ &= \sum_{g_0,\dots,g_4=g_0} \nu_1(g_0,g_2)\nu_1^*(g_1,g_2)\nu_1(g_1,g_3)\nu_1(g_3,g_4) \\ &= \sum_{\{g_i\}} \prod_{\langle ij\rangle} \nu_1^{s_{ij}}(g_i,g_j) \end{aligned} \tag{18}$$



where i, j are vertex order index.

where $s_{ij} = 1$ if edge $\langle ij\rangle$ has $+$ orientation
and $s_{ij} = -1$ or $*$ if edge $\langle ij\rangle$ has $-$ orientation

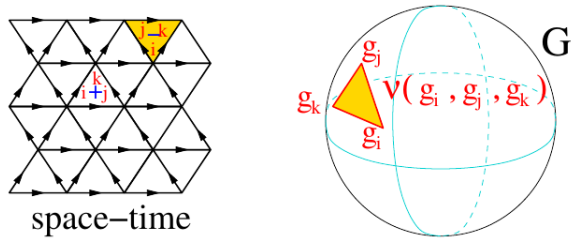
Lattice topological NLσM and group cohomology

Definition of Lattice topological NLσM by path integral

- Path integral on $1+1$D space-time lattice with branching (simplicial complex or triangularization) structure

$$e^{-S} = \prod \nu_2^{s_{ijk}}(g_i, g_j, g_k) \tag{19}$$

where $\nu_2^{s_{ijk}} = e^{-\int_\triangle \mathcal{L}_{fixed}}$ and $s_{ijk} = 1, *$



- The path integral above defines a LNLσM with target space G on $1+1D$ space-time lattice.
- The LNLσM has (global)symmetry G if for any $g_i, h \in G$, we have

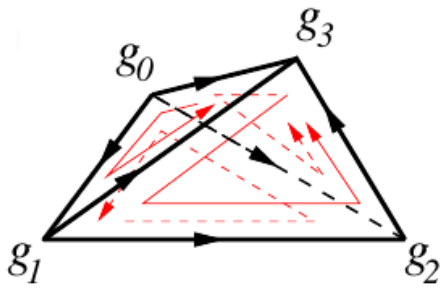
$$\nu_2(g_i, g_j, g_k) = \nu_2(hg_i, hg_j, hg_k) \tag{20}$$

Lattice NLσM and group cohomology

Definition of Lattice topological NLσM by path integral

In order to have Lattice topological NLσM,

- we need $e^{-S_{fixed}} = \prod \nu_2^{s_{ijk}} = 1$ on any triangularization of sphere



On a tetrahedron lattice we have 2-cocycle condition

$$\nu_2(g_1,g_2,g_3)\nu_2(g_0,g_1,g_3)\nu_2^{-1}(g_0,g_2,g_3)\nu_2^{-1}(g_0,g_1,g_2) = 1 \quad (21)$$

- $\nu_2(g_i,g_j,g_k) \in U(1)$ (this represents a quantum phase)
- The 2-cocycle condition has many solutions, we say two solution $\nu_2 \sim \tilde{\nu}_2$ if they satisfy

$$\tilde{\nu}_2(g_0,g_1,g_2) = \nu_2(g_0,g_1,g_2)\frac{\beta_1(g_1,g_2)\beta_1(g_0,g_1)}{\beta_1(g_0,g_2)} \quad (22)$$

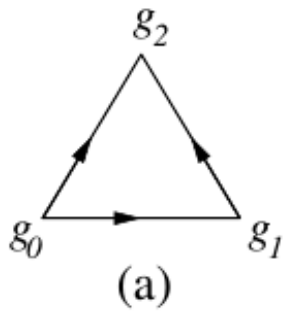
Lattice NLσM and group cohomology

Group cohomology

The extra factor

$$\frac{\beta_1(g_1,g_2)\beta_1(g_0,g_1)}{\beta_1(g_0,g_2)} = d\beta_1(\triangle) = \beta_1(\partial\triangle) \tag{23}$$

,with $\beta_1(g_i,g_j) = \beta_1(hg_i,hg_j)$, is nothing but the co-boundary value of the triangle.



(a)

The set of equivalent classes of ν_2 is denoted as group cohomology

$$\mathcal{H}^2(G,U(1)) \tag{24}$$

$\mathcal{H}^2(G,U(1))$ describes $1+1$D SPT phases protected by G, more general $\mathcal{H}^{d+1}(G,U(1))$ describes $d+1$D SPT phases protected by G

Lattice NLσM and group cohomology

Group cohomology and SPT phases

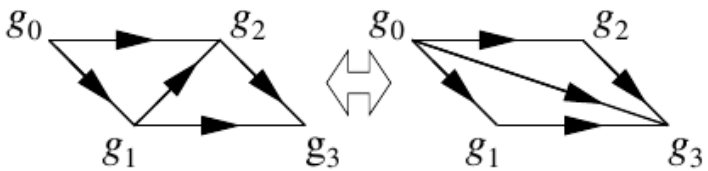
In order to show that $\mathcal{H}^2(G, U(1))$ describes $1+1$D SPT phases protected by G, we need to show

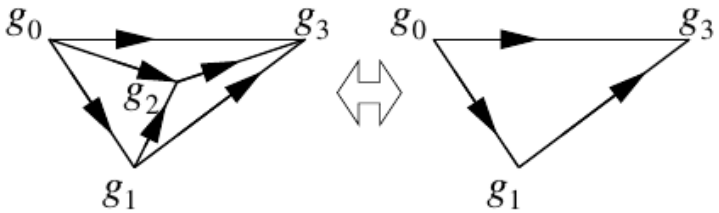
- The action that described by $\nu_2(g_i, g_j, g_k)$ is topological
- The action is symmetric under global transformation
- The ground state of this action is SRE state

The second condition is already satisfied by the condition on ν_2 and β_1. We can check the first condition by consider the different triangularizations of space time

Lattice NLσM and group cohomology

Group cohomology and SPT phases





We see that the action amplitude does not change when we change the triangularization since

$$\nu_2(g_0,g_1,g_2)\nu_2^{-1}(g_1,g_2,g_3)=\nu_2(g_0,g_1,g_3)\nu_2^{-1}(g_0,g_2,g_3) \tag{25}$$

$$\nu_2(g_0,g_1,g_2)\nu_2^{-1}(g_1,g_2,g_3)\nu_2(g_0,g_2,g_3)=\nu_2(g_0,g_1,g_3) \tag{26}$$

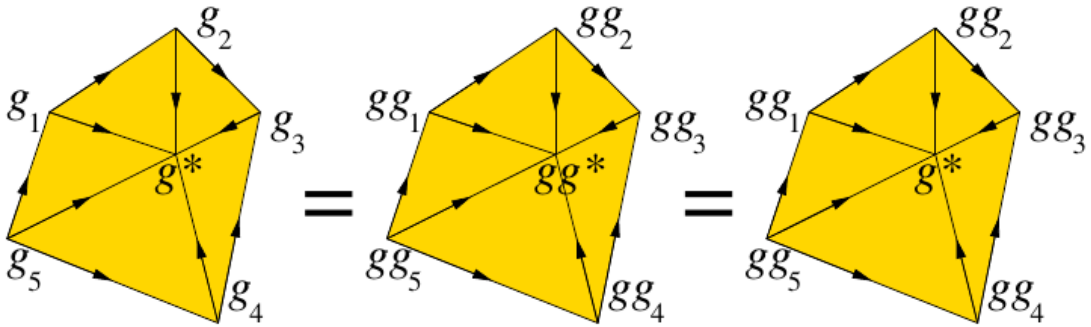
both of them come from cocycle condition

$$\nu_2(g_1,g_2,g_3)\nu_2(g_0,g_1,g_3)\nu_2^{-1}(g_0,g_2,g_3)\nu_2^{-1}(g_0,g_1,g_2)=1 \tag{27}$$

Thus the Lattice topological NLσM that we defined is a RG fixed point theory (no metric or scale dependence).

Lattice NLσM and group cohomology

Ground state of SPT phases

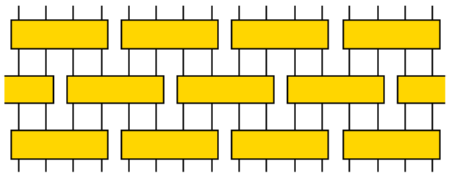


The ground state wave function $\Psi(\{g_i\}) = \prod_i \nu_2(g_i, g_{i+1}, g^*)$

- It is symmetric under the G-transformation
- It is equivalent to a product state $|\Psi_0\rangle = \otimes_i \sum_{g_i} |g_i\rangle$ under a LU transformation (note that $\Psi_0(\{g_i\}) = 1$)

$$\Psi(\{g_i\}) = \prod_{i=\text{even}} \nu_2(g_i, g_{i+1}, g^*) \prod_{i=\text{odd}} \nu_2(g_i, g_{i+1}, g^*)\Psi_0(\{g_i\})$$

$$= \quad \Psi_0(\{g_i\}) \rightarrow \text{Short-range entangled}$$



The ground state is symmetric with a trivial topo. order

Summation table of group cohomology

Symmetry G	$d=0$	$d=1$	$d=2$	$d=3$
$U_1 \rtimes Z_2^T$ (top. ins.)	$\mathbb{Z}$	$\mathbb{Z}_2$ (0)	$\mathbb{Z}_2$ $(\mathbb{Z}_2)$	$\mathbb{Z}_2^2$ $(\mathbb{Z}_2)$
$U_1 \rtimes Z_2^T \times$ trans	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}_2^8$
$U_1 \times Z_2^T$ (spin sys.)	0	$\mathbb{Z}_2^2$	0	$\mathbb{Z}_2^3$
$U_1 \times Z_2^T \times$ trans	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^9$
Z_2^T (top. SC)	0	$\mathbb{Z}_2$ $(\mathbb{Z})$	0 (0)	$\mathbb{Z}_2$ (0)
$Z_2^T \times$ trans	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$
U_1	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$U_1 \times$ trans	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^4$
Z_n	$\mathbb{Z}_n$	0	$\mathbb{Z}_n$	0
$Z_n \times$ trans	$\mathbb{Z}_n$	$\mathbb{Z}_n$	$\mathbb{Z}_n^2$	$\mathbb{Z}_n^4$
$D_{2h} = Z_2 \times Z_2 \times Z_2^T$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$
$SO(3)$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
$SO(3) \times Z_2^T$	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$

Appendix

Group cohomology

Define an element of 2^{nd}simplicial homology group in G as a map of a triangle (with specific order of vertices) to G
Since we see that $\nu_2(g_i, g_j, g_k)$ looks like the map from a element of 2^{nd} simplicial homology group of G to $U(1)$, we may guess $\nu_2(g_i, g_j, g_k)$ as element of cohomology group, in which the base space is group G itself and the coefficient ring is $U(1) = R/Z$.
However, if we look more careful, $\nu_2(g_i, g_j, g_j)$ only depends on the vertex g_i, g_j, g_k and the orientation. Therefore the group cohomology is not what we guessed, and it can not be thought as a cohomology group in general.