Holographic Entanglement Entropy (HEE)

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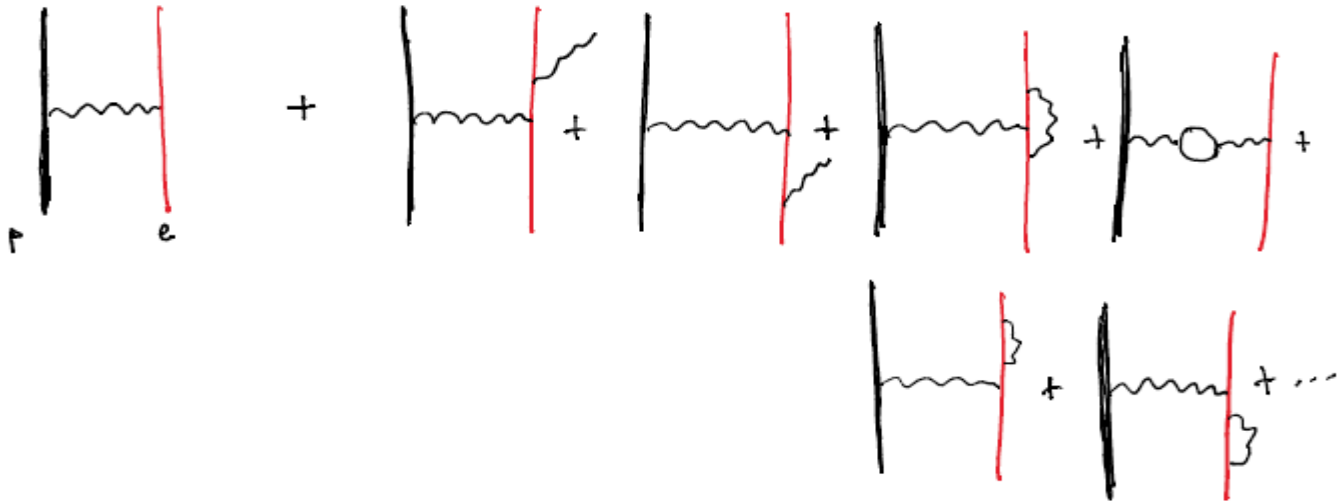
I. Introduction to Holography

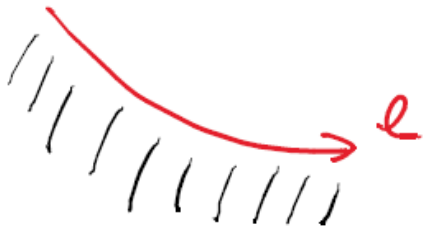
1. Motivation

- Electron scattering on heavy charged particle



- Effective Coulomb potential

$$V(r) = -\frac{\alpha}{r}\left(1 + \frac{\alpha}{4\sqrt{\pi}}\frac{e^{-2m_e r}}{(m_e r)^{3/2}} + \cdots\right)$$

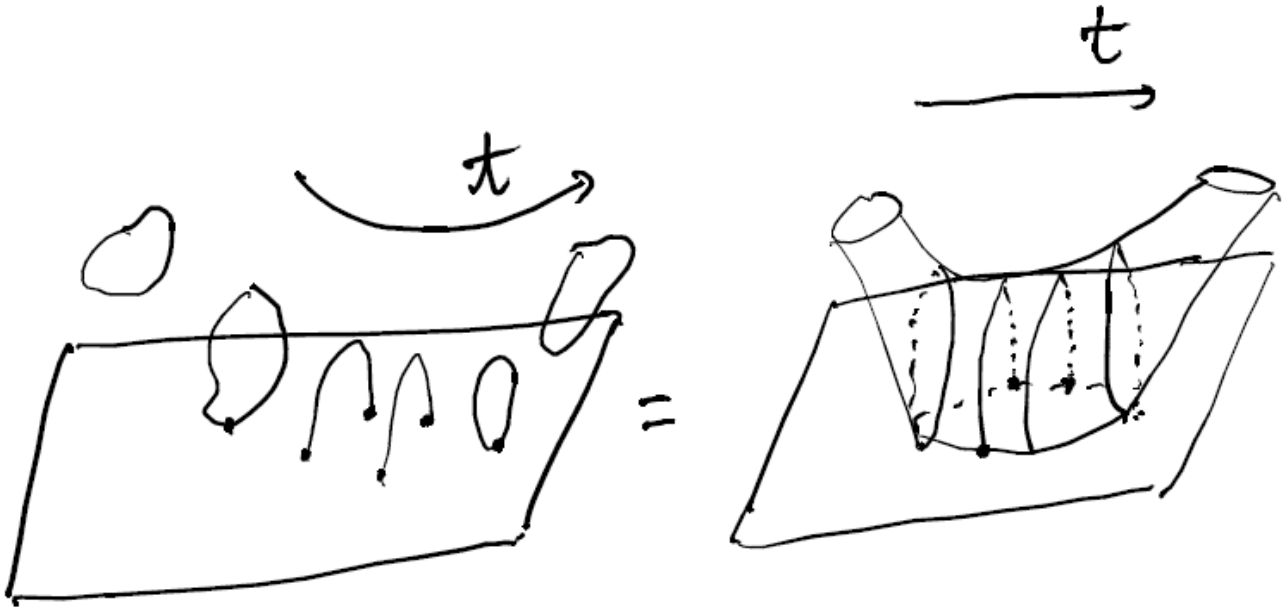


We have 2 approaches for this process

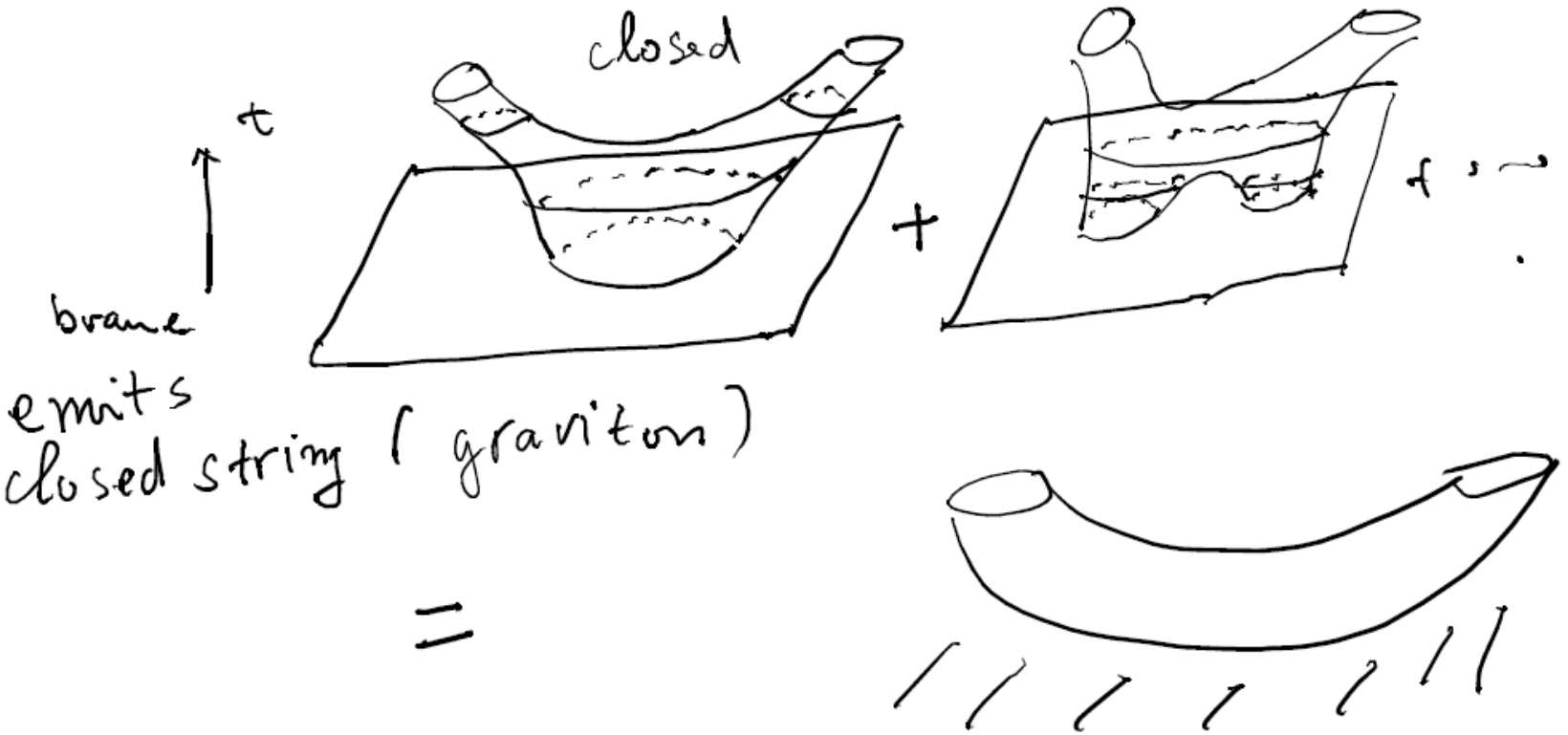
- (A): We can sum over all Feynman diagrams.

- (B): We can also consider the electron moves in specific background potential V(r)
- The similar situation happens when we consider the AdS/CFT duality. AdS/CFT is the different approaches of the same Physics.

- Consider the process of open string scattering on D-branes



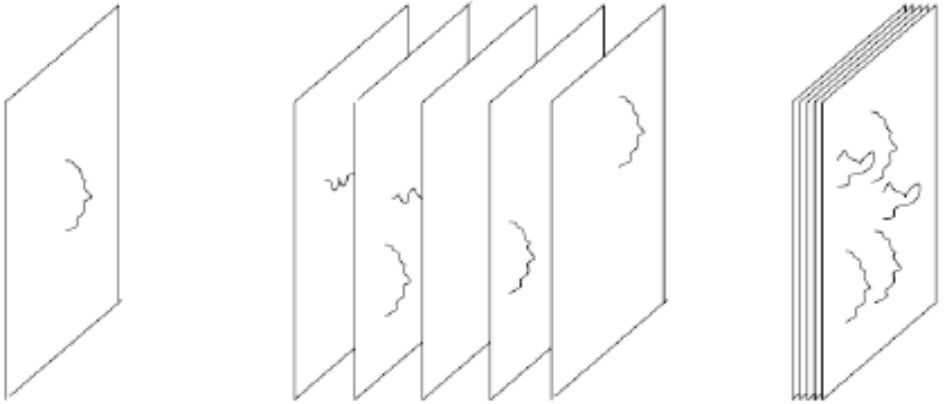
- We can consider a closed string scatters on D-brane and sum over all possible process including open strings (Approach A).



- Statement of open-closed duality.
- D-brane can emit closed string (in particular graviton) to make non-trivial background.
- We can consider process of an closed string moves in non-trivial background (Approach B).

- When we consider the open string picture (approach A), the D-branes' degree of freedom at low energy limit includes massless open strings which are gauge fields.
- In the case we have N D-branes stack together, we have U(N) Yang-Mill theory for D-branes' degree of freedom.

$U(1)^N \longrightarrow U(N)$



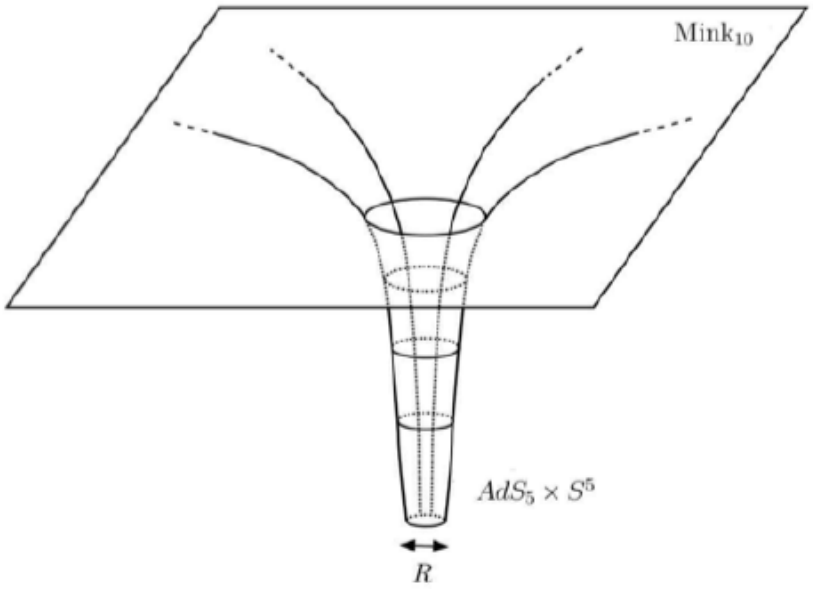
- From string theory, read off from the effective action for of D-branes, we have the relation

$$g_s \sim g_{YM}^2$$

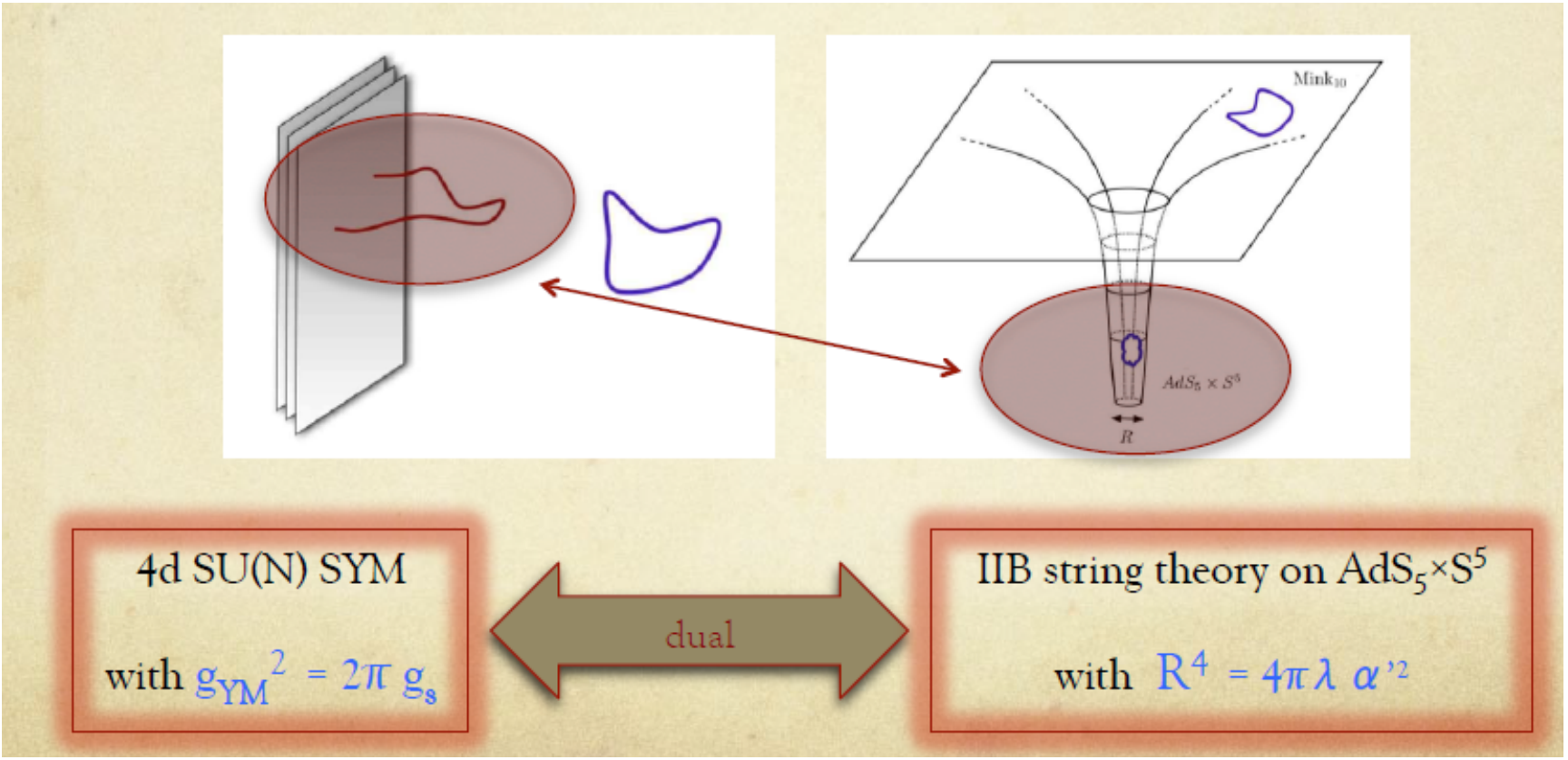
- We also can consider D-branes as the source of background (approach B) have the gravitational back ground induced by N D-branes

$$ds^2 = f^{-1/2}(-dt^2 + d\vec{x}^2) + f^{1/2}(dr^2 + r^2 d\Omega_5^2), \qquad f = 1 + \frac{R^4}{r^4}, \qquad R^4 = 4\pi g_s N \alpha'^2$$

- For $r >> R$ we get flat spacetime.
- For $r << R$ we get $AdS_5 \times S^5$ geometry: $ds^2 \approx \frac{R^2}{r^2} dr^2 + \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + R^2 d\Omega_5^2$



- In the low energy limit $E \ll 1/\sqrt{\alpha'} = 1/l_s$ or $\alpha' \to 0$



- In which we have 't Hoft parameter $\lambda = N g_{YM}^2 = N g_s$

- Consider 't Hoft limit which keeps λ finite but take $N \to \infty$ that gives us classical string theory $g_s \to 0$.

$$\frac{R^4}{l_p^4} \sim \frac{R^4}{\sqrt{G}} \sim N$$

- Consider the gravitational radius in compare with string length.

$$\frac{R^4}{\alpha'^2} = 4\pi\lambda \qquad and \qquad g_s = \frac{\lambda}{2\pi N}$$

- $O(\alpha')$ corrections $\longleftrightarrow$ $O(\lambda^{-1/2})$ corrections
- $O(g_s)$ corrections $\longleftrightarrow$ $O(N^{-1})$ corrections (at fixed λ)

- In large λ limit, we have classical gravity in the AdS side. In small λ limit, we have perturbation gauge theory.

AdS/CFT conjecture

- The statement is the equality of the partition function of two theories. CFT lives in the boundary of AdS.

$$\int_{\phi|_{\partial}=\bar{\phi}} D\phi(z,\vec{x})\, e^{-S[\phi(z,\vec{x})]} \equiv Z_{AdS}[\bar{\phi}(\vec{x})] = Z_{CFT}[\bar{\phi}(\vec{x})] \equiv \left\langle e^{\int d^d x \bar{\phi}(\vec{x}) O(\vec{x})} \right\rangle$$

Boundary conditions in AdS are associated with sources for the field operators.

- Which we can use to compute correlation functions of operators

$$\left\langle O_{\Delta_1}(x_1) O_{\Delta_2}(x_2) \ldots \right\rangle = \left. \frac{\partial^n Z[\phi_{\Delta_i}]}{\partial \bar{\phi}_{\Delta_1}(x_1) \partial \bar{\phi}_{\Delta_2}(x_2) \ldots} \right|_{\bar{\phi}_{\Delta_i}=0}$$

II.Holographic EE conjecture

- General holographic principle:

D+2 dimensional quantum gravity on M

= D+1 non-gravitational theory on ∂M

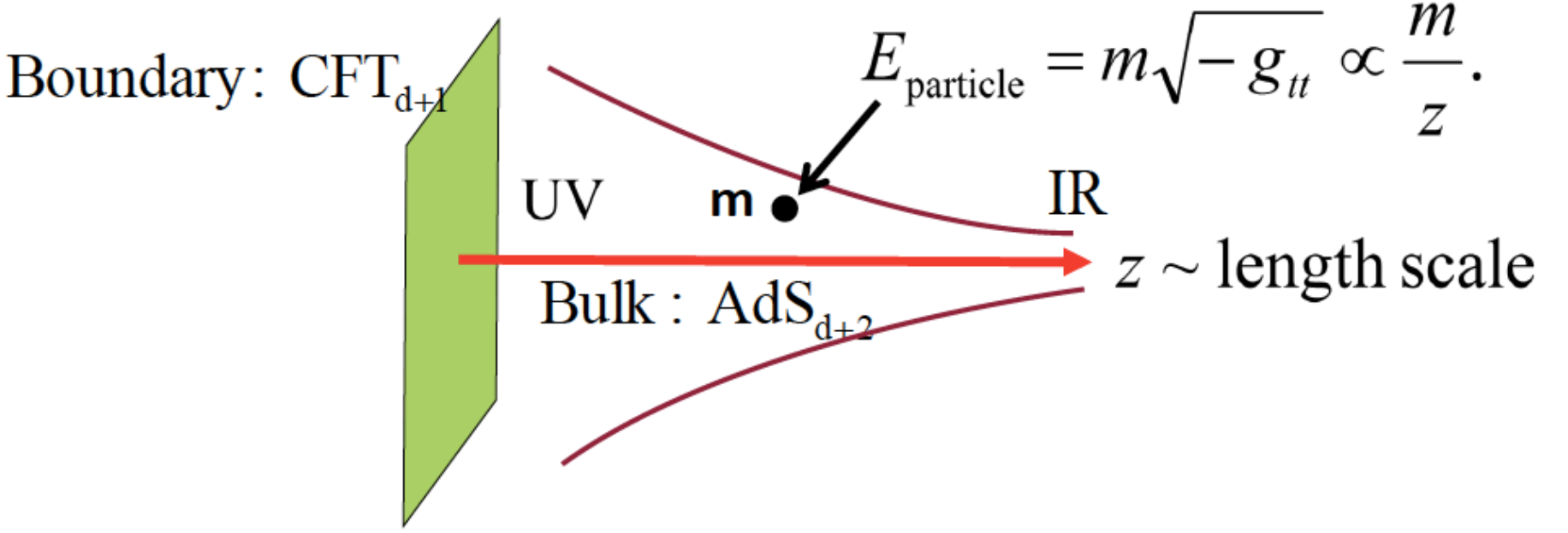
- Entropy of Black Hole $\sim$ area of horizon in Planck units

$$S_{BH} = \frac{A}{4G_N}$$

- In which Newton constant in D+2 dimension: $G_N \sim l_p^D$

Interpretation of the extra dimension

$$\text{AdS metric}: \quad ds^2 = R^2 \cdot \frac{dz^2 - dt^2 + d\vec{x}^2}{z^2}$$



The extra dimension z can be consider as length scale. The cut-off in EE can be consider as AdS space with condition $z > \epsilon$

Ryu-Takayanagi conjecture

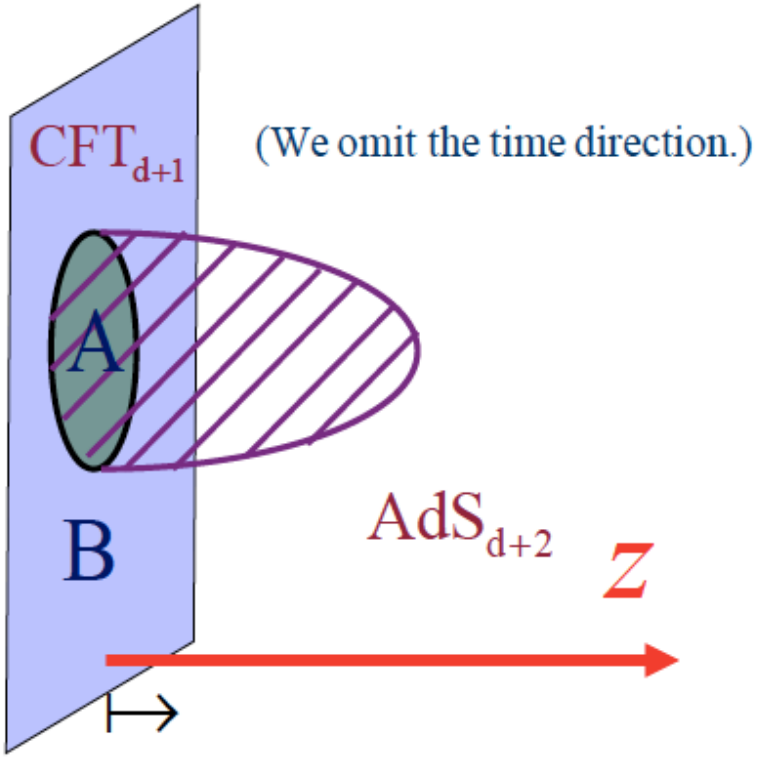
Consider CFT (which has gravity dual theory that can be described by classical Einstein gravity) with subsystem A

$$S_A = \underset{\substack{\partial\gamma_A=\partial A \\ \gamma_A \approx A}}{\mathrm{Min}} \left[\frac{\mathrm{Area}(\gamma_A)}{4G_N} \right]$$

γ_A is the minimal area surface (codim.=2) such that

$$\partial A = \partial \gamma_A \text{ and } A \sim \gamma_A \ .$$

homologous



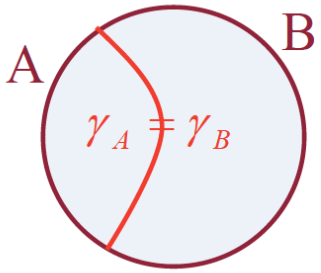
Note that we used Euclidean signature

$$ds^2 = R^2 \frac{-dt^2 + \sum_{i=1}^{d} dx^i dx^i + dz^2}{z^2}$$

Some remarks

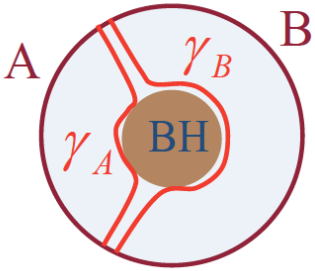
- The HEE conjecture works in static back ground metric. So we can consider the minimal surface in a time slice.
- The minimal surface γ_A has to be homologous to the sub-system A. That means there is a bulk region r_A such that

$$\partial(r_A) = A \cup \gamma_A$$



(Pure) AdS $\Leftrightarrow$ Zero temp. CFT

ρ_{tot} is pure $\Leftrightarrow$ $S_A = S_B$.



AdS BH $\Leftrightarrow$ Finite temp. CFT

ρ_{tot} is not pure $\Leftrightarrow$ $S_A \neq S_B$.

- Ryu-Takayanagi conjecture was proved by Lewkowycz and Maldacena in 2013 using AdS/CFT conjecture.

- From the choice of minimum area γ_A with the cut-off $\epsilon \ll R$. We see that the main contribution to Area(γ_A) is the part closed to the boundary $z \approx \epsilon$

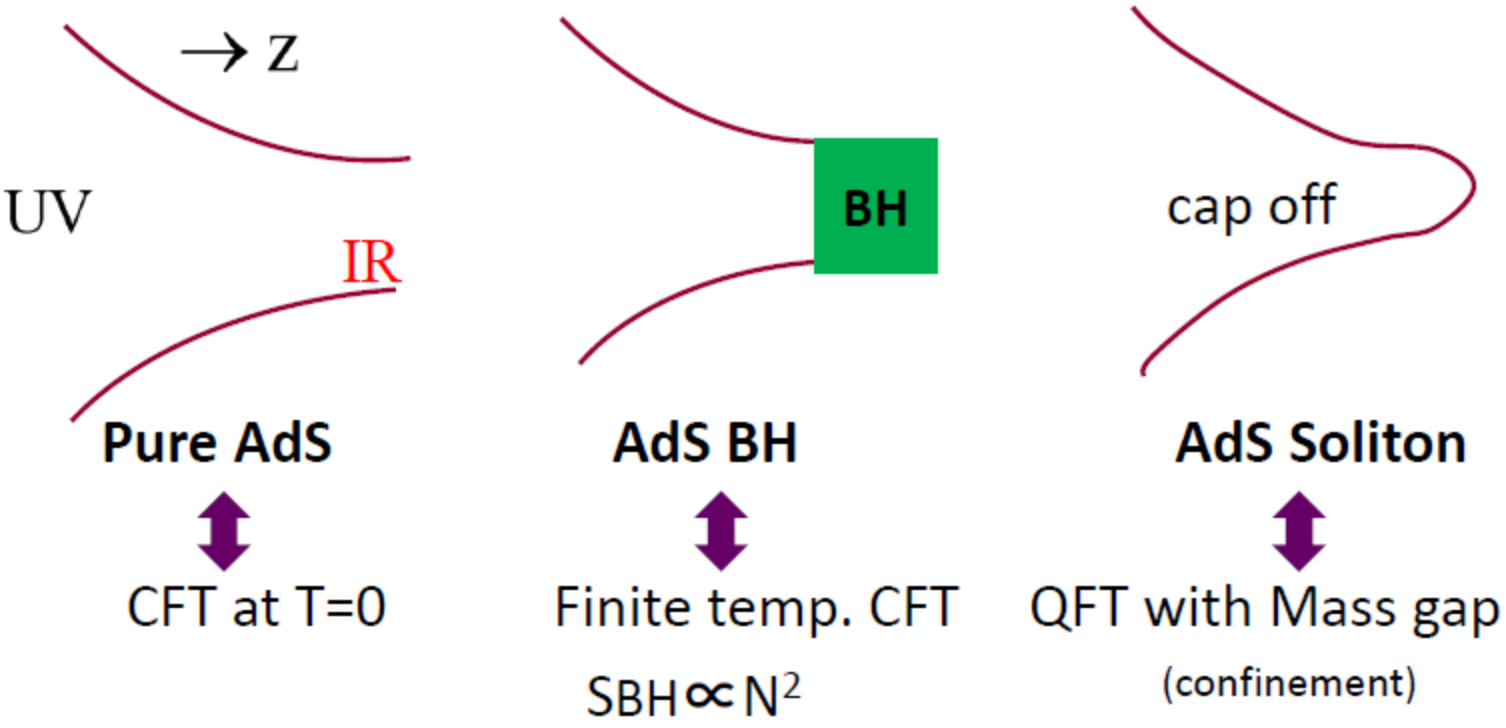
$$Area(\gamma_A) \sim \frac{Area(\partial\gamma_A)}{\epsilon^{d-1}} + Subleading\ terms$$

- We also have conditior $\partial\gamma_A = \partial A$
- Thus we recover the Area law of EE

$$S_A \sim \frac{Area(\partial A)}{\epsilon^{d-1}} + Subleading\ terms$$

- The gravity side has to be described by classical gravity, Therefore, the CFT theory should be in strong coupling and large N limit.

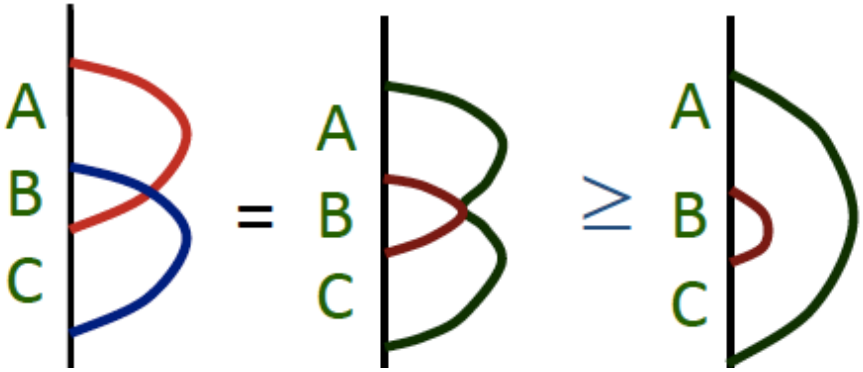
- We also can deform the AdS to get the specific dual QFT with fixed points.



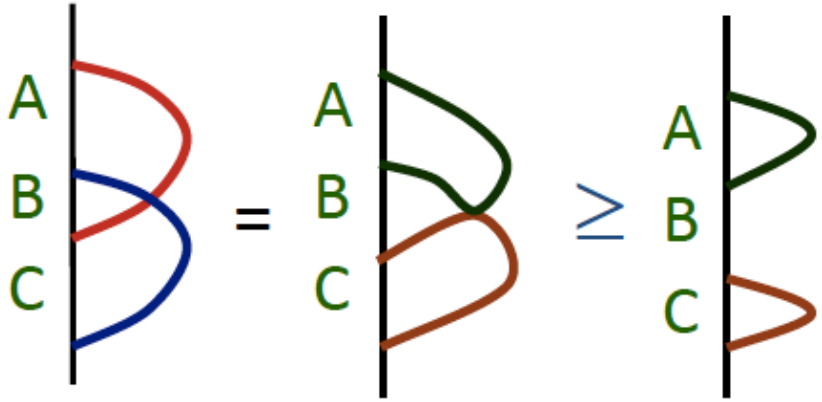
- We will consider the finite temperature case in the examples.

III. Holographic proof of EE inequalities

- With the HEE formula, we can easily prove strong subaddicity using geometric argument.



$$\Rightarrow S_{A+B}+S_{B+C}\geq S_{A+B+C}+S_{B}$$

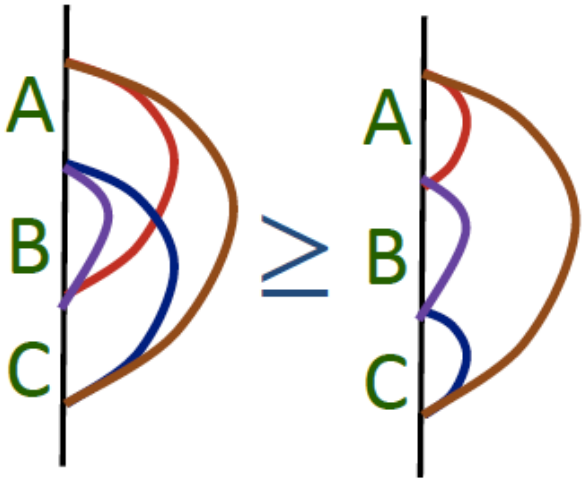


$$\Rightarrow S_{A+B}+S_{B+C}\geq S_{A}+S_{C}$$

Bonus inequality

- The HEE has extra inequality (called monogamy)

$$S_{AB}+S_{BC}+S_{AC}\geq S_A+S_B+S_C+S_{ABC}$$
$$\Leftrightarrow I(A:B)+I(A:C)\leq I(A:BC)$$



- The last inequality only works in the case that QFT has dual gravity theory (e.x. Large N gauge theories). This inequality is not true for all QFT.

IV. HEE calculation examples

- The most trivial example is AdS_{1+2}/CFT_{1+1}

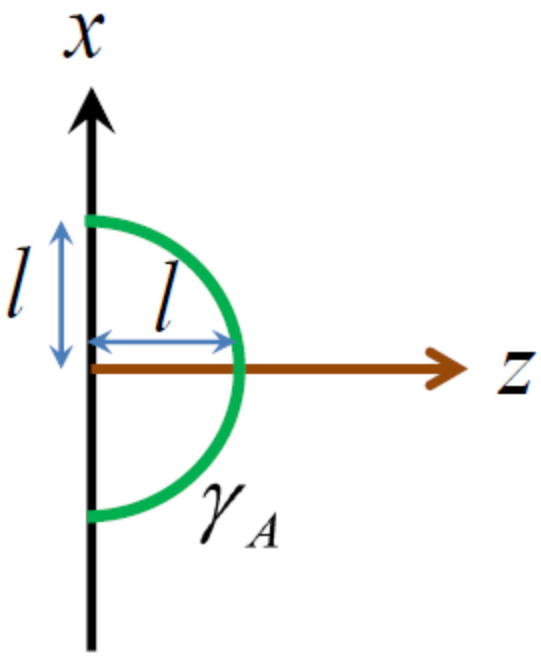
$$ds^2 = R^2 \frac{-dx_0^2 + \sum dx_i^2 + dz^2}{z^2}.$$

- The central charge of the dual CFT

$$c = \frac{3R}{2G_N^{(3)}}$$

- The minimum length solution x(z)

$$x(z) = \sqrt{l^2 - z^2}$$



$$length(\gamma_A) = \int \sqrt{g_{ab} dx^a dx^b} = 2R \int_\epsilon^l dz \frac{2l}{\sqrt{l^2 - z^2}} = 2R log \frac{2l}{\epsilon}$$

- We have HEE

$$S_A = \frac{length(\gamma_A)}{4G_N^{(3)}} = \frac{c}{3} log \frac{2l}{a}$$

- This is the exact result of EE in 1+1 CFT.

- We consider finite temperature EE, which is equivalent to insert a black hole into gravity side. We calculate EE of straight belt in finite temperature.
- Black hole metric

$$ds^2 = -\frac{r^2}{R^2} f(r) dt^2 + \frac{r^2}{R^2} d\vec{x}^2 + \frac{R^2}{r^2} \frac{dr^2}{f(r)}$$

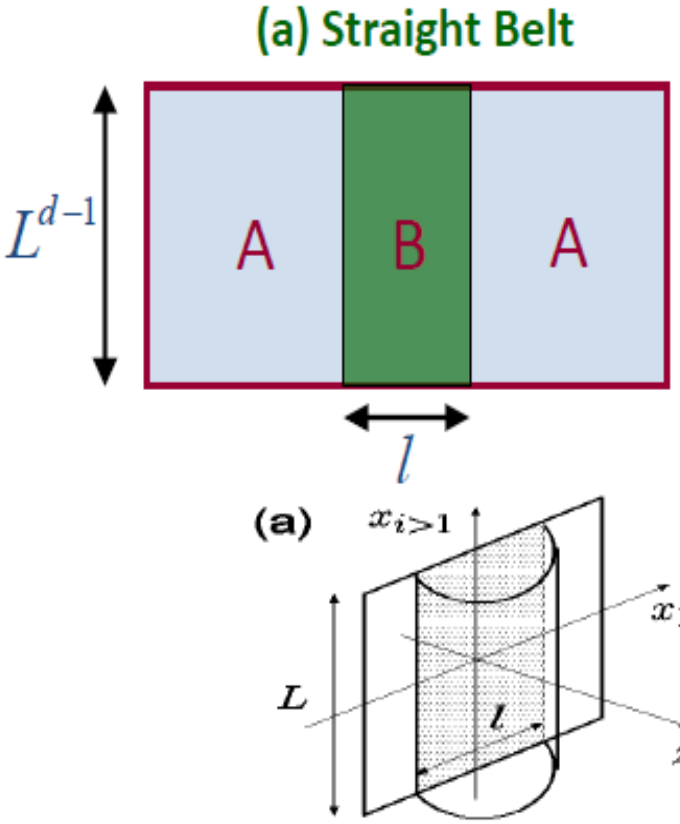
$$f(r) = 1 - \frac{r_H^{D+1}}{r^{D+1}}$$

- It is asymptotic AdS metric with

$$z = \frac{R^2}{r}$$

- Consider straight belt

$$X \equiv x^1 = \left[-\frac{l}{2}, \frac{l}{2}\right], \; x^i = \left[-\frac{L}{2}, \frac{L}{2}\right], \; i = 2, ..., D$$



- We have Hawking temperature

$$T = \frac{r_H (D+1)}{4\pi R^2}$$

- We have to minimize the area

$$Area(\gamma_B) = \frac{2L^{D-1}}{R^{D-1}} \int_0^{l/2} dX r^{D-1} \sqrt{\frac{r^2}{R^2} + \frac{R^2 \left(dr/dX\right)^2}{r^2(1 - \frac{r_H^{D+1}}{r^{D+1}})}}$$

- We can consider it is the action with X is time coordinate. We have Hamiltonian

$$H = \frac{\partial \mathcal{L}}{\partial(dr/dX)} \frac{dr}{dX} - \mathcal{L} = -\frac{2L^{D-1}}{R^{D-1}} \frac{r^{D+1}}{R^2 \sqrt{\frac{r^2}{R^2} + \frac{R^2 r^{D-1} (dr/dX)^2}{r^{D+1} - r_H^{D+1}}}}$$

- The Lagrangian doesn't depend on time explicitly, thus the energy is conserved. At X=0, we have

$$r(0) = r_c$$

$$H = -\frac{2L^{D-1}}{R^{D-1}} \frac{r^{D+1}}{R^2\sqrt{r^2 + \frac{r^{D-1}(dr/dX)^2}{r^{D+1}-r_H^{D+1}}}} = -\frac{2L^{D-1}r_c^D}{R^2}$$

- Thus we can read off the equation

$$\frac{dX}{dr} = \pm \frac{R^2 r_c^D}{r^{D+2}\sqrt{\left(1-\frac{r_c^{2D}}{r^{2D}}\right)\left(1-\frac{r_H^{D+1}}{r^{D+1}}\right)}}$$

- With boundary condition

$$X(\infty) = \pm\frac{l}{2}$$

- The BC gives us

$$\frac{l}{2} = \int_{r_c}^{\infty} \frac{R^2 r_c^D}{r^{D+2}\sqrt{\left(1-\frac{r_c^{2D}}{r^{2D}}\right)\left(1-\frac{r_H^{D+1}}{r^{D+1}}\right)}} dr$$

- By substitution

$$u = \frac{r_C}{r}$$

- We get

$$\frac{l}{2} = \frac{R^2}{r_C}\int_0^1 \frac{u^D du}{\sqrt{1-u^{2D}}}\left(1-\frac{r_H^{D+1}}{r_c^{D+1}}u^{D+1}\right)^{-1/2}$$

- Further more we can rewrite the area formula

$$Area(\gamma_B) = \frac{2L^{D-1}}{R^{D-1}}\int_{r_C}^{\frac{R^2}{\epsilon}} dr r^{D-1}\sqrt{\frac{r^2}{R^2}\frac{dX}{dr} + \frac{R^2}{r^2(1-\frac{r_H^{D+1}}{r^{D+1}})}}$$

- When we consider AdS_{1+2}/CFT_{1+1} , in which D=1. The above integral can be calculated in closed form.

$$r_C = r_H Coth\left(\frac{l r_H}{2R^2}\right)$$

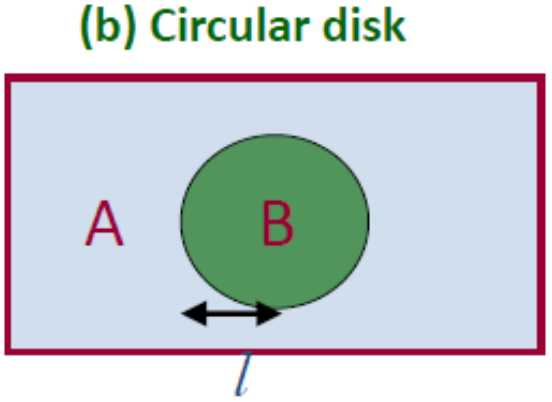
- The final answer for HEE

$$S_B = \frac{c}{3} log\left[\frac{\beta}{\pi\epsilon} sinh\left(\frac{\pi l}{\beta}\right)\right] \qquad \beta = \frac{1}{T} \qquad c = \frac{3R}{2G_N^{(3)}}$$

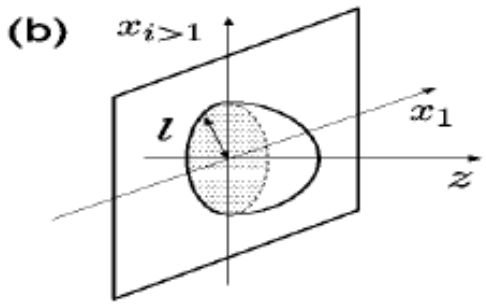
- Which is also compatible to the EE result in CFT.

- Consider HEE of circle disk, in which sub system A is d-dim disk of radius l. We consider polar coordinates on the boundary.

$$dx_i^2 = dr^2 + r^2 d\Omega_{d-1}^2$$



- By symmetry, the minimal area surface is defined by z(r).



$$Area(\gamma_B) \sim R^D \int dr d\Omega \sqrt{detG_{ij} dx^i dx^j} = R^D vol(S^{D-1}) \int_0^l dr r^{D-1} \frac{\sqrt{1 + \left(\frac{dr}{dz}\right)^2}}{z^d}$$

- Euler Lagrangian equation for z(r)

$$rz\frac{d^2z}{dr^2} + (D-1)z\left(\frac{dz}{dr}\right)^3 + (D-1)z\frac{dz}{dr} + Dr\left(\frac{dz}{dr}\right)^2 + Dr = 0$$

- Boundary condition z(l)=0.

$$r^2 + z^2 = l^2$$

- Using substitution $y = \frac{l}{z}$, put back the cut off

$$Area(\gamma_B) = R^D vol(S^{D-1}) \int_{\epsilon/l}^{1} dy \frac{(1-y^2)^{(D-2)/2}}{y^D}$$

$$= \frac{2\pi^{D/2}R^D}{\Gamma(D/2)}\left[\frac{1}{D-1}\left(\frac{l}{\epsilon}\right)^{D-1} - \frac{D-2}{2(D-3)}\left(\frac{l}{\epsilon}\right)^{D-3} + \ldots\right]$$

- The Entanglement Entropy by the conjecture

$$S_B = \frac{2\pi^{D/2}R^D}{4G_N^{(D+2)}\Gamma(D/2)} \int_{\epsilon/l}^{1} dy \frac{(1-y^2)^{(D-2)/2}}{y^D}$$

$$= p_1(l/\epsilon)^{D-1} + p_3(l/\epsilon)^{D-3} + \begin{cases} p_{D-1}(l/\epsilon) + p_D + \mathcal{O}(\epsilon/l) & D\ even \\ P_{D-2}(l/\epsilon)^2 + qlog(l/\epsilon) + \mathcal{O}(\epsilon/l) & D\ odd \end{cases}$$