Intro:

QFT on Curved Spacetime

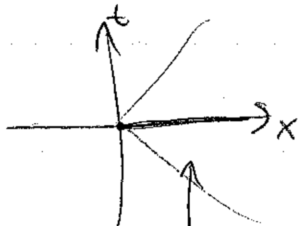
· **Plan**

- lightning review of QFT on flat spacetime for real free KG field.
- General construction for QFTs on curved spacetime (free)
  - expose ambiguity in approach: notion of "positive frequency" ⟹ ambiguity in notion of particles.

  (in flat spacetime, we could get around this since inertial observers all agree on what's positive frequency & what isn't)

  (in curved space, inertial observers not available to you

  - ambiguity remains in flat space:
- Application: uniformly accelerated observers in flat spacetime.
  - observe thermal bath of particles



only have access to this region

$$\rho_R = \mathrm{Tr}_L |\Omega_M\rangle\langle\Omega| = e^{-\frac{2\pi}{a} H_\eta}$$ ← "modular hamiltonian"

where H_η is generator of boosts

- Won't have time to get to, but this fact has many applications in holographic information theory
  - Casini Huerta Myers 1102.0440
  - Nozaki, Numasawa, Prudenziati, Takayanagi 1304.71

Quantizing Free Real Scalar in Flat Space

- Recall: Expand the field operator in fourier modes

$$\phi(x) = \sum_{k^2=-m^2} \left(a_k e^{ik\cdot x} + a_k^\dagger e^{-ik\cdot x} \right)$$

$$\Rightarrow \Box\phi = m^2 \phi$$

- CCR $\Rightarrow [a_k, a_{k'}^\dagger] = \delta_{kk'}$
- by virtue of this, have a fock space decomposition of the many body hilbert space $\mathcal{H}$.

$N_k = a_k^\dagger a_k$ complete set of commuting, hermitian operators

$\mathcal{H} = \bigoplus_{n_k} \mathcal{H}_{n_k}$ ↑ hilbert space w/ n particles in mode k.

- unique vacuum: $a_k|0\rangle_a = 0$ state w/ "no particles"

Ambiguities

(the one choice from which everything followed)

- made a choice of basis for the solution space (complexified).
- many possible choices: $\{f_i, f_i^*\}$

$$\phi(x) = \sum_i (b_i f_i^* + b_i^\dagger f_i)$$

- New fock space decomposition arising from $N_i^b = b_i^\dagger b_i$ & a new vacuum $b_i|0\rangle_b = 0$.
- in general, b_i will include $a^\dagger$'s & so

$b_i|0\rangle_a \neq 0$. $|0\rangle_a$ "has particles".

(← will generally happen when new notion of positive freq. "the f's" involves both positive & negative freq's in the a construction).

QFT on Curved Spacetime

- real, free KG field (all considerations will generalize to arbitrary free theories.)

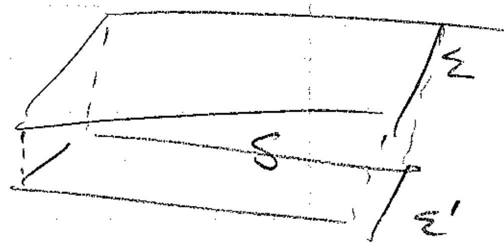
- $\exists$ antisymmetric, nondegenerate form on $\mathcal{S}$ (the solution space). the <u>symplectic form</u>

$$\Omega(\phi_1,\phi_2) = \int_\Sigma (\phi_1 n^\mu \nabla_\mu \phi_2 - \phi_2 n^\mu \nabla_\mu \phi_1)$$

n^μ is unit normal to Σ., $\Omega(\phi_1,\phi_2) = -\Omega(\phi_2,\phi_1)$

- well-defined on $\mathcal{S}$ by virtue of the EOM.

$$\Omega_\Sigma(\phi_1\phi_2) - \Omega_{\Sigma'}(\phi_1,\phi_2) = \int_{\partial S}(\phi_1 n^\mu \nabla_\mu \phi_2 - \phi_2 n^\mu \nabla_\mu \phi_1)$$
$$= \int_S \nabla_\mu(\phi_1 \nabla^\mu \phi_2 - \phi_2 \nabla^\mu \phi_1) = 0.$$



- Want to construct a Hilbert space of classical solutions
- Start w/ $\mathcal{S}^{\mathbb{C}}$ & define

$$\langle \phi_1,\phi_2\rangle = i\Omega(\phi_1^*,\phi_2)$$

properties: (i) $\langle\,,\rangle$ is nondegenerate since Ω is.
(ii) $\langle\phi_1,\phi_2\rangle^* = \langle\phi_2,\phi_1\rangle$ a hermitian form.
(iii) $\langle\phi_1^*,\phi_2^*\rangle = -\langle\phi_2,\phi_1\rangle$

problem: not positive definite $\langle f,f\rangle > 0 \Rightarrow \langle f^*,f^*\rangle < 0$.

- Can always choose an orthonormal basis $\{f_i, f_i^*\}$

$$\langle f_i,f_j\rangle = -\langle f_j^*, f_i^*\rangle = \delta_{ij}$$

- $\mathcal{H}_1 = \text{Span}\{f_i\}$ (the cauchy completion of)

Single particle hilbert space $(\mathcal{H}_1\langle\,,\rangle)$

Now proceed w/ previous story

$$\phi = \sum_i (a_i f_i^* + a_i^\dagger f_i) \qquad \text{vacuum state } a_i|0\rangle_a = 0.$$

If made another choice of basis $\{g_i, g_i^*\}$, would have

$$\phi = \sum_i (b_i g_i^* + b_i^\dagger g_i) \qquad b_i|0\rangle_b = 0$$

How are these choices related, by taking inner product w/ basis vectors.

$\langle f^*_i, \phi \rangle$: $\quad a_i = \sum_j (c_{ij} b_j + d_{ij} b^+_j)$

$\langle g^*_i, \phi \rangle$: $\quad b_i = \sum_j (a_{ij} a_j + b_{ij} a^+_j)$

where $\quad c_{ij} = \langle g_j, f_i \rangle, \quad d_{ij} = \langle g^*_j, f_i \rangle$

$\qquad a_{ij} = c^*_{ji} \qquad b_{ij} = -d^*_{ji}$

- called bogoliubov coefficients

- finding these is simply down to evaluating an integral, what is relationship between the two vacua in terms of them?

$$\langle N^b_i \rangle_a = {}_a\langle 0 | b^+_i b_i | 0 \rangle_a = \sum_j |b_{ij}|^2$$

{this is fact mentioned before, if positive & negative freq. modes mixed $\langle g^*_j, f_i \rangle \neq 0$, then a vacuum has particles from the b perspective

(can be more explicit)

- Can find explicit formula for a vacuum in terms of b vacuum.

Expand $\quad |0\rangle_a = \sum_{n=0}^{\infty} \psi_{i_1 \cdots i_n} b^+_{i_1} \cdots b^+_{i_n} |0\rangle_b$

↑ n-particle wavefunction

Solve $a_i |0\rangle_a = 0 \implies (b_i - \varepsilon_{ij} b^+_j)|0\rangle_a = 0$ where $\varepsilon_{ij} = -\sum_k c^{-1}_{ik} d_{kj}$

Now insert vacuum:

$$\psi_i |0\rangle_b + \sum_{n=1} \left((n+1) \psi_{i i_1 \cdots i_n} - \varepsilon_{i i_1} \psi_{i_2 \cdots i_n} \right) b^+_{i_1} \cdots b^+_{i_n} |0\rangle_b = 0$$

- recursion relation for n-particle wavefunction in terms of n-2 particle one:

$$\psi_{i_1 \cdots i_{2n+1}} = 0 \qquad \psi_{i_1 \cdots i_{2n}} = \frac{1}{(2n)!!} \varepsilon_{i_1 i_2} \cdots \varepsilon_{i_{2n-1} i_{2n}}$$

(Exactly solves for $|0\rangle_a$ in terms of $|0\rangle_b$ & the bogoliubov coefficients

Rindler Spacetime

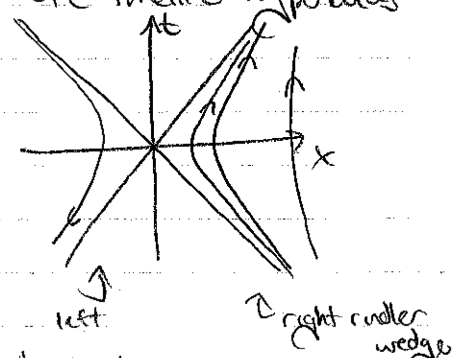
(Now let's apply this formalism to a problem)

- seek particle content of $|0\rangle_M$ according to uniformly accelerated observers.

Consider a patch of Minkowski covered by

$$\begin{pmatrix} t \\ x \end{pmatrix} = \pm \begin{pmatrix} a^{-1}e^{a\xi}\sinh a\eta \\ a^{-1}e^{a\xi}\cosh a\eta \end{pmatrix} \qquad -\infty < \xi, \eta < \infty$$

Note: $-t^2+x^2 = a^{-1}e^{2a\xi}$ so curves of const ξ are timelike hyperbolas



for such curves $\sqrt{a_\mu a^\mu} = ae^{-a\xi}$
(so as $\xi \to \infty$ observers approach inertial, as $\xi \to -\infty$, accel diverges.)

- say this is coord system adopted by observer of accel a.

- note existence of horizon, observers in R have no access to info in L & vice versa.

- but both R & L are globally hyperbolic spacetimes in own right
  - well posed initial value formulation on each wedge, can consider classical mech in L or R & construct corresponding hilbert space

For simplicity, take $m=0$ case.

EOM reads $(-\partial_\eta^2 + \partial_\xi^2)\phi = 0$

positive freq. sols in right & left rindler wedges

$${}^R g_k = \begin{cases} \left(\frac{\pi}{\omega}\right)^{\frac{1}{2}} e^{-i(\omega\eta - k\xi)} & \text{in } R \\ 0 & \text{in } L \end{cases} \qquad {}^L g_k = \begin{cases} \left(\frac{\pi}{\omega}\right)^{\frac{1}{2}} e^{i(\omega\eta - k\xi)} & \text{in } L \\ 0 & \text{in } R \end{cases}$$

positive freq. since η runs backwards.

$\omega = |k|$

Expand field

$$\phi = \int \frac{dk}{2\pi} \frac{1}{\sqrt{2|k|}} \left(a_k e^{i(\omega t - kx)} + a_k^\dagger e^{-i(\omega t - kx)} \right)$$

-or- $\phi = \int \frac{dk}{2\pi} \left({}^R b_k \, {}^R g_k^* + {}^R b_k^\dagger \, {}^R g_k + \text{left} \right)$

- in principle it remains to ~~[illegible]~~ evaluate some inner products to get $|0\rangle_M$ in terms of $|0\rangle_b$.

- Much easier way: can prove

$${}^R g_k + e^{-\pi\omega/a} ({}^L g_k)^* \quad \text{is purely positive (fourier) frequency}$$

$$\therefore \left({}^R b_k + e^{-\pi\omega/a} b_k^\dagger \right) |0\rangle_a = 0$$

can read off the ζ matrix

$$\zeta_{ij} = \begin{cases} e^{-\pi\omega/a} \delta_{ij} & \text{when one mode is in } R \text{ + one in } L \\ 0 & \text{otherwise} \end{cases}$$

Gives $\quad |0\rangle_M = \prod_i \sum_{n_i=0}^{\infty} e^{-\pi n_i \omega_i/a} |n_i\rangle_L \otimes |n_i\rangle_R \quad$ maximally entangled state.

where $|n_i\rangle_R = \frac{1}{\sqrt{n_i!}} ({}^R b_i^\dagger)^{n_i} |0\rangle_R$ etc...

Can now retrieve modular hamiltonian.

$$\rho_R = \mathrm{Tr}_L |0\rangle_M {}_M\langle 0| = \prod_i \sum_{n=0}^{\infty} e^{-2\pi n_i \omega_i/a} |n_i\rangle_R {}_R\langle n_i| = e^{-\frac{2\pi}{a} H_R}$$

where H_R is the generator of boosts.

$$H_R = \int \frac{dk}{2\pi} \omega_k b_k^\dagger b_k.$$

Actually, this is a particular case of a more general fact from axiomatic QFT:

Bisognano-Wichmann theorem

$$\rho = e^{-2\pi H_\eta} \qquad H_\eta \text{ generator of boosts.}$$