Weak Localization Inherited from Classical Chaos

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ABSTRACT: This note is a contribution to Kadanoff Center for Theoretical Physics journal club meeting in 2017 spring quarter. In this talk, we follow the seminal works by Aleiner and Larkin [1–3], discussing how the divergence between classically chaotic trajectories leads to corrections to conductivity.

Contents

1 Introduction

In the very beginning of the quarter we had an introduction to classical chaos, where we briefly introduced some quantitative measures in that context. What followed was the quantum corrections in semi-classical limit, dubbed quantum chaos, which heavily replied on some properties in classical dynamical systems such as the existence of periodic orbits, etc.

However, during the discussion of quantum chaos, we have not seen the legacy like Lyapunov exponent from classical problems.

In this talk, the Lyapunov exponent strikes back in terms of the correction to conductivity. In usual semi-classical regime, $\lambda_{\mathrm{F}} \ll \ell_{\mathrm{tr}}$ where ℓ_{tr} is the transport mean-free path defined as the product of Fermi velocity and transport relaxation time. Depending on the scale of scatterer a, problems can be further grouped into 2 subsets. If the uncertainty of momentum p_{F} direction is $\delta\theta$, we know the diffraction spreading is $\delta\theta = \lambda_{\mathrm{F}}/a$. The uncertainty in position is given by $\delta x \simeq \ell_{\mathrm{tr}}\delta\theta = \ell_{\mathrm{tr}}\lambda_{\mathrm{F}}/a$. If δx is (much) smaller than a, treating the motion in terms of classical trajectory still works. The quantum feature appearing in this regime is called quantum chaos. On the other hand, when δx is much larger than a, the motion is described by diffusion process and the regime is dubbed quantum disorder.

In short, $\ell_{\mathrm{tr}}\lambda_{\mathrm{F}} < a^2$: Quantum Chaos; $\ell_{\mathrm{tr}}\lambda_{\mathrm{F}} > a^2$: Quantum Disorder.

As we will see shortly, for 2 dimensional systems, in the former regime another time scale, the Ehrenfest time t_{E}, matters and it turns out the divergence of classical trajectory provides a correction of the same order as weak localization.

The plan of the talk is the following. After the introduction, we review the idea of weak localization and its correction to conductivity in 2 space dimensions. Standard references include [4, 5]. Next we dive into Aleiner and Larkin's work [1–3], and derive the correction with similar approaches to those in section 2. The physical meaning and related works follow to close the talk and note.

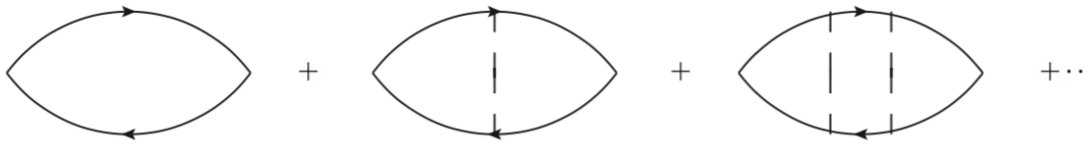

Figure 1. The leading contributions to the current-current correlation functions.

2 Weak Localization

As most people can be admit that one of the important issues in hard condensed matter theory is calculating transport coefficients such as electric and thermal conductivities. The most primitive model, proposed even earlier than the birth of quantum theory, was the celebrated Drude model. It turns out this model indeed captures some essence of transport physics. The insight of this model is abstracted in terms of the time scale τ.

From standard many-body point of view, the conductivity tensor can be calculated using Kubo formula.

$$\sigma^{ij} = -\frac{1}{\nu_n}\Big[\langle J^i(\nu')J^j(-\nu')\rangle\Big]_{\nu'=0}^{\nu'=i\nu_n}, \tag{2.1}$$

where ν_n is the Matsubara frequency and the ensemble average contains a bunch of diagrams. The leading order, from figure 1 contributes to Drude conductivity with τ being the quasiparticle scattering time.

$$\sigma^{ij} = \frac{ne^2}{m}\frac{1}{\tau^{-1} - i(i\nu_n)}\delta^{ij}. \tag{2.2}$$

Summing the ladder diagrams renormalizes τ with $\tau_{\rm tr}$, the transport scattering time. They are not always the same since backward scatterings are the main players who relax the current. Small angle scatterings, which contribute to τ, do not necessary make the current decay.

The ladder diagrams effective renormalize the vertex. Even if we include interaction to replace bare electron propagators with dressed version. The final form still looks pretty much the same with mass replaced with renormalized mass.

We note that there is another set of diagrams, crossing diagrams in figure 2, is not accounted for. These diagrams are of order $\lambda_{\rm F}/\ell_{\rm tr}$ (the ratio between Fermi wavelength and mean-free path) smaller than ladder diagrams (owing to the allowed phase space around Fermi surface) and therefore are not taken care of in the very beginning, yet I will provide some arguments to relate them to the weak-localization correction to the conductivity.

The computation leading to (2.2) and its renormalize version corresponds to Born approximation, in which a single scattering event takes place on one impurity at a time. The interference between scattering on different impurities is never incorporated in this scheme. Such an approximation is valid

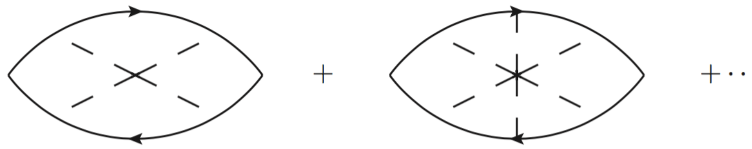

Figure 2. The cross diagrams that are not included in leading order calculations for electronic conductivity.

when those distinct paths are not so coherent. Taking the interference of 2 paths for example. According to Feynman, given an initial state and a final state, to calculate the probability $P_{f\leftarrow i}$ quantum mechanically we have to first compute the total amplitude $\mathcal{A} = \mathcal{A}^{(1)}_{f\leftarrow i} + \mathcal{A}^{(2)}_{f\leftarrow i}$. Then

$$P_{f\leftarrow i} = |\mathcal{A}|^2 = P^{(1)}_{f\leftarrow i} + P^{(2)}_{f\leftarrow i} + 2\sqrt{P^{(1)}_{f\leftarrow i}P^{(2)}_{f\leftarrow i}}\cos\phi, \tag{2.3}$$

where ϕ is the phase angle between 2 amplitudes. The Born approximation takes advantage of the fact that for non-coherent waves the fluctuation interference does not contribute. However, a very special case is the return probability. Regarding this problem, the starting and the end points are the same. For each closed path, we find its time-reversed cousin. Quite intuitively, the quantum probability calculated for this pair of paths is twice as large as the corresponding classical probability, that $P_{\rm qm} = 4P_{f\leftarrow i} > 2P_{f\leftarrow i} = P_{\rm cl}$, since they are basically coherent, implying $\phi = 0$. In short, quantum interference enhances the probability for a particle returning to its original position. In terms of transport property, the correction to electrical conductivity should reduce the Drude value.

This is a heuristic argument for weak localization, following which we explain why those coherent time-reversed path pairs are corresponding to those cross diagrams. Referencing Coleman, suppose in such a scattering process the particle has encountered n scattering processes. The amplitude can be represented by a product of retarded Green's function G_R,

$$\mathcal{A}^{(1)} = G_R(n, n-1)G_R(n-1, n-2)...G_R(2, 1), \tag{2.4}$$

while its time-reversed partner has the amplitude

$$\mathcal{A}^{(2)} = G_R(1, 2)G_R(2, 3)...G_R(n-1, n). \tag{2.5}$$

Instead of using ϕ, the interference term can also be written as $\mathsf{Re}[\mathcal{A}^{(2)*}\mathcal{A}^{(1)}]$. Recall that in Fourier space $G_R(2-1,\omega)^* = G_A(2-1,\omega)$. Thus,

$$\mathsf{Re}[\mathcal{A}^{(2)*}\mathcal{A}^{(1)}] = \prod_{i=1}^{n-1} G_R(j+1, j; \omega)G_A(j+1, j : \omega), \tag{2.6}$$

corresponding to the Cooperon diagrams in figure 3.

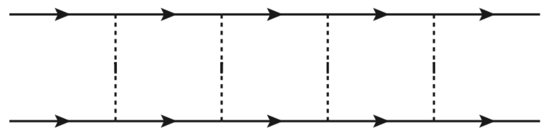

Figure 3. The interference terms form the so-called Cooperon diagrams.

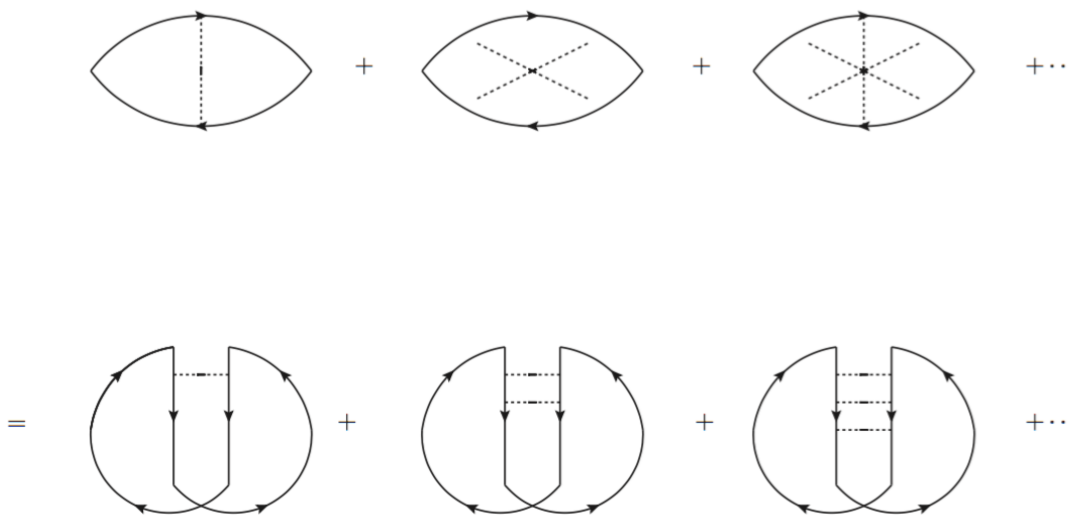

Figure 4. By twisting Feynman diagrams, we see the sum over cross diagrams corresponds to evaluating of Cooperon diagrams.

Next we look back at the cross diagrams. We see in principle we can twist everyone of them without changing the topology such that each one contains a Cooperon ladder. See figure 4. We hence confirm coherent interference is encoded in cross diagrams.

Borrowing the argument in P. Phillips, we can estimate the functional behavior of $\frac{|\delta\sigma|}{\sigma}$. The key to contribution from closed loop trajectory is phase coherence. Therefore we introduce a time scale τ_ϕ after which the particle loses it phase coherence. Within the coherence time, we look at the ratio between 2 volumes. One is the space that the particle can travel via diffusion process, and the other is the spacetime volume of the wave packet. The idea is if the particle still lies in the latter volume after τ_ϕ, it contributes to the closed-loop return probability. The diffusion volume is $(\sqrt{D_0 t})^d$, while the coherent volume is $\lambda_{\mathrm{F}}^{d-1} v_{\mathrm{F}} \mathrm{d}t$. These give the estimate

$$\frac{|\delta\sigma|}{\sigma} = \int_{\tau}^{\tau_\phi} \frac{v_{\mathrm{F}} \lambda_{\mathrm{F}}^{d-1} \mathrm{d}t}{(D_0 t)^{d/2}}. \tag{2.7}$$

As we can see, this correction depends strongly on dimensionality. For $d = 2$, we have the correction

$\delta\sigma \propto -\log\frac{\tau_\phi}{\tau}$. To be precise

$$\delta\sigma = -\frac{1}{2\pi^2}\frac{e^2}{\hbar}\log\frac{\tau_\phi}{\tau}. \tag{2.8}$$

The logarithmic divergence indicates the electrons are eventually localized in 2 dimensions. Note that we mentioned the weak localization correction can be derived using standard diagrammatic technique. In Fourier space, the correction (in coherent limit) can be written as

$$\delta\sigma = -\frac{e^2}{2\pi^2\hbar}\log\Big(\frac{1}{\omega\tau_{\rm tr}}\Big)\Gamma(\omega), \omega\tau_{\rm tr} \leq 1, \tag{2.9}$$

where $\Gamma(\omega)$ is a renormalization function. It is known that in quantum disorder regime, it equals 1 regardless of frequency and therefore is *universal*. In the next section we shall "derive" this function in quantum chaotic regime and show it is non-trivial at finite frequency.

3 Correction to Weak Localization in Chaotic Regime

3.1 A New Scale

In the previous discussion we look at semi-classical picture in the regime where $\lambda_{\rm F} \ll \ell_{\rm tr}$, while the scattering events are led by point-like impurities. The weak localization is discussed at the level of diffusive process. We could imagine the wave packet of size $\lambda_{\rm F}$ at this diffusive localization regime started from a smaller wave packet whose size is much smaller than the scatters. The consideration introduces another time scale, the Ehrenfest time $t_{\rm E}$. The theme of the talk here, as I would say, is the suppression of weak localization before $t_{\rm E}$. It turns out the underlying key to this correction is the exponential separation between trajectories with close initial conditions in classical chaotic systems. Therefore, the Lyapunov exponent comes into play.

To address this problem, let us first try to look at how the diffusion process may go wrong. One of the assumptions of diffusion equation is that electrons lose their memory. Even if during their journeys they may have encountered a scatterer more than once, those scattering events are considered independent because they mostly revisit that scatterer with different momenta.

However, similar to last section, there are some trajectories related by time-reversal can provide coherent interference and thus must be correlated. To be specific, suppose a trajectory starts with initial data $\mathbf{r}(0)$ and $\mathbf{p}(0)$. After some time T, the particle arrives at $\mathbf{r}(T)$ and $\mathbf{p}(T)$ subject to conditions $|\mathbf{r}(T) - \mathbf{r}(0)| = \rho_0 \ll a$ and $\mathbf{p}(0) = -\mathbf{p}(T)$. The motions of the particle in the final state $[\mathbf{r}(T-t), \mathbf{p}(T-t)]$ and the initial stage $[\mathbf{r}(t), \mathbf{p}(t)]$ are correlated and therefore diffusive description breaks down. For an illustration, see figure 5.

Here we can raise a heuristic argument to see how a new time scale would emerge. Instead of the same trajectory, looking at the region, the Lyapunov region, where first several scattering events take place and diffusive description is not applicable, the initial and final stages can be regarded as 2 trajectories with $[\mathbf{r}(t), \mathbf{p}(t)]$ and $[\mathbf{r}(T-t), -\mathbf{p}(T-t)]$. Denoting the directions of momenta by $\mathbf{n}$, we know that $\mathbf{r}(0) \simeq \mathbf{r}(T)$ and $\mathbf{n}(0) = \mathbf{n}_i = -\mathbf{p}(T) = -\mathbf{n}_f$ cannot hold exactly simultaneously because of

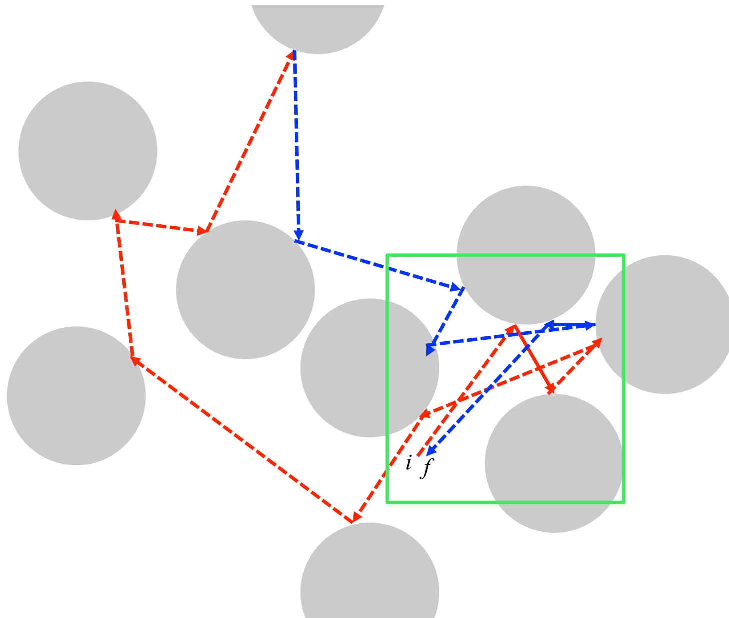

Figure 5. In this figure we depict a class of paths whose initial and final stages are strongly correlated. In the green box, red trajectory (outgoing) and blue trajectory (incoming) feel the same potential and the diffusive description does not apply here.

uncertainty principle. A typical estimate for ρ is λ_{F}. The uncertainty in direction $\delta\phi = |\mathbf{n}_f \times \mathbf{n}_i|$, in terms of which the path difference in this region is of order $\delta\phi^2 a$. Therefore, $\delta\phi$ is the order of the square root of the wave spread $\delta\phi = \sqrt{\lambda_{\mathrm{F}}/a}$. In the chaotic system, such uncertainty is exponentially amplified $\delta\phi(t) = \delta\phi_0\, e^{\lambda t}$. Besides, we know at certain time $t^* < T/2$ $\delta\phi$ is of order 1 since at $T/2$ trajectories close. Thus we have $T \simeq \frac{2}{\lambda} \log \delta\phi^{-1}$. Together with $\delta\phi \simeq \sqrt{\lambda_{\mathrm{F}}/a}$, we obtain the Ehrenfest time scale

$$t_{\mathrm{E}} = \frac{1}{\lambda} \log \frac{a}{\lambda_{\mathrm{F}}}. \tag{3.1}$$

Let us do try to calculate the correction using the idea of probability distribution as in last section. First we define the relative position and momentum as

$$\boldsymbol{\rho}(t) = \mathbf{r}(t) - \mathbf{r}(T - t) \tag{3.2}$$

$$\mathbf{k}(t) = \mathbf{p}(t) + \mathbf{p}(T - t), \tag{3.3}$$

which satisfy the equations of motion

$$\frac{\partial \boldsymbol{\rho}}{\partial t} = \frac{\mathbf{k}}{m} \tag{3.4}$$

$$\frac{\partial \mathbf{k}}{\partial t} = \frac{\partial U[\mathbf{r}(T - \mathbf{r})]}{\partial \mathbf{r}} - \frac{\partial U[\mathbf{r}(t)]}{\partial \mathbf{r}}. \tag{3.5}$$

We would like to calculate the probability distribution $W_0(T, \rho_0)$ for a particle satisfying $|\mathbf{r}(T) - \mathbf{r}(0)| = \rho_0$ and $\mathbf{p}(T) = -\mathbf{p}(0)$. This can be separated into 2 parts. The first part $\mathrm{d}t\, W(a, \rho_0; t)$ is the conditional probability that given $\rho(0) = \rho_0$, $\rho(t) > a$ within $[t, t + \mathrm{d}t]$. This part is responsible for the particle leaving the Lyapunov region. The other part, not surprisingly, should be responsible for it returning after the diffusive motion $W_{\mathrm{D}}(a, t)$, referring to the probability that the particle approaches its starting point within the range a. The probability is than the convolution.

$$W_0(T, \rho_0) = \int_0^T \mathrm{d}t\, W_{\mathrm{D}}(a, T - 2t) W(a, \rho_0; t). \tag{3.6}$$

The factor of 2 should be understood. If the particle spends t to get out of Lyapunov region, as it returns, it needs another t to follow path similar to the original one.
First we look at $W(a, \rho_0; t)$, or more generally $W(\rho, \rho_0; t)$ with $a > \rho > \rho_0$. Expanding (3.5) to linear order in ρ,

$$\frac{\partial k_j}{\partial t} = -\rho_i \hat{\mathcal{M}}_{ij},\ \hat{\mathcal{M}}_{ij} = \frac{\partial^2 U}{\partial r_i \partial r_j}. \tag{3.7}$$

From either visual inspection or (3.4) we see the linearization implies $\mathbf{r}ho$ also has exponential time dependence. In the simplest case where $\hat{\mathcal{M}}$ is stationery, we know $\rho(t) \simeq \rho(0)e^{\lambda t}$. We call λ, the most negative eigenvalue of $\hat{\mathcal{M}}$, the Lyapunov exponent in this context. Nonetheless, in general the system cannot be solved exactly and thus extra insight or assumption is needed. Here we imagine that after the particle enters diffusive region, especially when $t \gg \tau_{\mathrm{tr}}$, the variation $\delta\lambda$ is random.

$$\frac{\mathrm{d}\log\rho}{\mathrm{d}t} = \lambda + \delta\lambda(t) \tag{3.8}$$

with $\langle \delta\lambda(t_1)\delta\lambda(t_2)\rangle = \lambda_2 \delta(t_1 - t_2)$. This is a Langevin equation. Solving the Fokker-Planck equation gives

$$W(\rho, \rho_0; t) = \left(\frac{\lambda^3}{2\pi\lambda_2 \mathcal{L}(\rho)}\right)^{1/2} \exp\left[-\frac{\lambda(\mathcal{L}(\rho) - \lambda t)^2}{2\lambda_2 \mathcal{L}(\rho)}\right], \tag{3.9}$$

where $\mathcal{L}(\rho) = \log \rho/\rho_0$. It is commented that this result is based on physical consideration, though specific λ and λ_2 can only be calculated for some special models. We also note that in order for the model not modeled by the tail of distribution, typically we need $T \geq \mathcal{L}(a)/(2\lambda)$, which is equivalent to the vicinity of the maximum of the distribution function $|\log|\rho/\rho_0| - \lambda t| \leq \lambda t$.
Next we consider W_{D}. Putting $W_0(T, \rho_0)$ in Fourier space by $\int \frac{\mathrm{d}\omega}{2\pi} W_0(\omega, \rho_0) e^{-i\omega T}$, the function $W_{\mathrm{D}}(a, \omega)$ and be further decomposed into 2 parts. One is responsible for the diffusion from scale $a \to \ell_{\mathrm{tr}}$, while the other is responsible for the diffusion process at the scale larger than ℓ_{tr}, where the standard diffusion equation works. At the time scale larger than τ_{tr}, the former is of order $1 + \mathcal{O}(\omega\tau_{\mathrm{tr}})$. The latter is given by the standard diffusion solution. Thus,

$$W_{\mathrm{D}}(\omega, a) = \frac{1}{4\pi D} \log \frac{1}{\omega t_{\mathrm{tr}}}, \tag{3.10}$$

together with which,

$$W_0(\omega, \rho_0) = W_{\mathrm{D}}(\omega, a) \exp\left(\frac{2i\omega\mathcal{L}(a)}{\lambda} - \frac{2\omega^2\lambda_2\mathcal{L}(a)}{\lambda^3}\right). \tag{3.11}$$

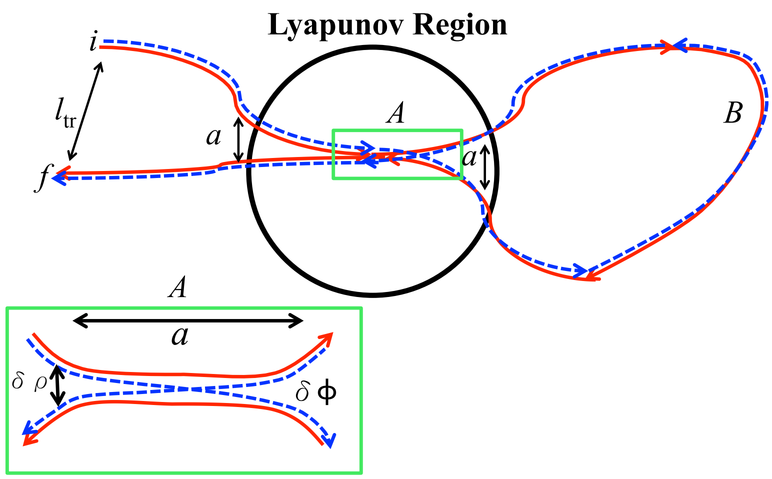

Figure 6. In this figure we illustrate the pair of paths that contribute to constructive interference in quantum chaos. The red solid path and blue dashed path are almost the same. The difference between them lies only in the green box region, where the red path grazes itself, and the blue path crosses itself. Thus, in journey B part they are essentially time-reversal pairs. The uncertainty is characterized by the distance $\delta\rho$ and the angle $\delta\phi$.

3.2 Interference Strikes Back

After all these consideration, however, we have been looking at a classical problem where diffusive description is not entirely applicable. Again according to Mr. Feynman, to have a quantum description, we have to sum over all possible directions of momentum, which then possibly washes out the effort above.

The day is saved again by finding some coherent paths which give constructive interference. Let us consider a starting and a final point i and f and the transport properties between them. Macroscopically the typical scale between i and f is of order $\ell_{\rm tr}$. Given 2 paths whose arc length difference is larger much than $\lambda_{\rm F}$, their oscillatory contribution vanishes and only classical part survives. Among those irrelevant paths, there are a class of pairs which are coherent. They are depicted in figure 6.

As we can see from the figure, since the red and blue paths almost coincide each other, to ensure coherence, we would like to discuss the constraint on $\delta\phi$ and the separation $\delta\rho$, which are subject to the uncertainty principle $\delta\phi\,\delta\rho = \lambda_{\rm F}$. In the figure, we see the arc length difference between 2 paths in the Lyapunov region $\delta\ell \simeq a\delta\phi^2$ and $\delta\rho \simeq a\delta\phi$. Demanding the former to be less than $\lambda_{\rm F}$ gives $\delta\phi \simeq \sqrt{\lambda_{\rm F}/a}$ and $\delta\rho = \sqrt{a\lambda_{\rm F}}$. Due to the size of $\delta\rho$, we see the non self-crossing path still almost grazes itself.[1]

[1]In the argument here, we use the length scale a instead of $\ell_{\rm tr}$, which is used in their original work. I chose a to keep my following argument and derivation self-consistent. Besides, in real experiments in ballistic cavity, a and $\ell_{\rm tr}$ are often of

Turning to the transport coefficient, we argue that order of $\delta\sigma$ should be proportional to the probability of finding the trajectories discussed above. To calculate that we can utilize the distributions W (3.9) and W_0 (3.11).

The journey starting and ending around the grazing point can be represented by

$$\mathrm{d}P_1 = \delta\rho\,\delta\phi\,v_{\mathrm{F}}\,\mathrm{d}t_1\,W_0(t_1, \sqrt{\lambda_{\mathrm{F}}a}), \tag{3.12}$$

and the probability for the particle leaving the Lyapunov region is $\mathrm{d}P_2 = \mathrm{d}t_2 W(a, \sqrt{\lambda_{\mathrm{F}}a}, t_2)$. Accounting for the fact that the total time took is $2t_2 + t_1$, in frequency domain

$$\frac{\delta\sigma}{\sigma} \simeq -\int \mathrm{d}P_1\,\mathrm{d}P_2 \simeq -\frac{1}{\pi\hbar\nu}\,W(a, \sqrt{\lambda_{\mathrm{F}}a}, 2\omega)W_0(\sqrt{\lambda_{\mathrm{F}}a}, \omega), \tag{3.13}$$

where we have rewrite $v_{\mathrm{F}}\lambda_{\mathrm{F}} = \frac{\hbar}{m} = \frac{1}{\pi\hbar\nu}$ using the 2 dimensional density of state (per spin) ν. In terms of W and W_{D}, the W products can be written as

$$W_{\mathrm{D}}W(a, \sqrt{\lambda_{\mathrm{F}}a}, 2\omega)^2 = \frac{1}{4\pi D}\log\frac{1}{\omega t_{\mathrm{tr}}}\,\exp\left(\frac{4i\omega\mathcal{L}(a)}{\lambda} - \frac{4\omega^2\lambda_2\mathcal{L}(a)}{\lambda^3}\right). \tag{3.14}$$

The logarithmic contribution corresponds to conventional weak localization. The numerical factor $\sigma/(4\pi^2\hbar\nu D)$ is simplified by Einstein relation $\sigma = 2e^2\nu D$. The latter exponent factor is identified as

$$\Gamma(\omega) = \exp\left(2i\omega t_{\mathrm{E}} - 2\frac{\omega^2\lambda_2 t_{\mathrm{E}}}{\lambda^2}\right),\ t_{\mathrm{E}} = \frac{1}{\lambda}\log\frac{a}{\lambda_{\mathrm{F}}}. \tag{3.15}$$

(2.9) is recovered. As we claimed, in quantum chaos regime, the universality of weak localization exhibits in DC limit, yet at finite frequency, there is a non-trivial renormalization due to Ehrenfest time.

3.3 Return of the Localization

As a final remark, in (2.8) we include coherence length, which encodes the time scale after which the paths lose coherence owing to inelastic scattering events, while (2.9) and the correction in quantum chaos regime do not. Here we state without proof that when one accounts for τ_ϕ,

$$\delta\sigma = -\frac{e^2}{2\pi^2\hbar}\sqrt{\Gamma(\omega)\Gamma(\omega + i\tau_\phi^{-1})}\log\frac{\tau_{\mathrm{tr}}^{-1}}{\sqrt{\omega^2 + \tau_\phi^{-2}}}, \tag{3.16}$$

which has a DC limit

$$\delta\sigma = -\frac{e^2}{2\pi^2\hbar}\exp\left(-\frac{t_{\mathrm{E}}}{\tau_\phi}\left[1 - \frac{\lambda_2}{\lambda^2\tau_\phi}\right]\right)\log\frac{\tau_\phi}{\tau_{\mathrm{tr}}}. \tag{3.17}$$

From this formula we then see a clear interpretation of the Ehrenfest time. The weak localization effect is suppressed before the wave packet spreads off over a length scale of the size of the scatterer. This case can take place in the studies of electronic transport in quantum dot. Moreover, the other fact that we assumed throughout the argument is time-reversal symmetry. This assumption can be loosen by turning on a magnetic field. Naturally another length scale, the magnetic length, comes into play and introduces other correction factors.

the same order.

4 Conclusion

To conclude, in this talk we try to point out the role of classical chaos plays in the correction to weak localization. To that end, we review the idea of weak localization and emphasize on the picture of interference between coherent classical trajectories, relating the correction to those cross diagrams or Cooperon diagrams.

Next we discuss special paths to which diffusive prescription is not applicable. In order to analyze those paths and their contribution to weak localization beyond diffusive story, we argue the large separation of initially close trajectories in classical chaotic problems turns out to play a role here, and therefore the Lyapunov exponent appears in the renormalization function.

At the end we briefly comment that the coherence time can also be included in the correction, and gives the result a rather intuitive interpretation that the weak localization is suppressed if the coherence time scale is smaller than the Ehrenfest time. We note that the quantities here can also be calculated using the semi-classical approaches together with Landauer formula. The formulation and derivation can be found in [6].

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