On-shell Recursion Relations of Scattering Amplitudes at Tree-level

Notes for Journal Club by Yikun Wang

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We will introduce the on-shell recursion relations of scattering amplitudes at tree-level today (in this note). One could use such recursion relations to calculate higher-point scattering amplitudes from only input of lower-point amplitudes at tree level. The idea used to build up the recursion relations is to use the analytic properties of on-shell scattering amplitudes, which will be extended to complex plane by defining proper complex shifts of the external momenta. The contents are based on Elvang and Huang's book [1].

Here is the structure of the talk (notes):

- Complex shits and the general recursion relation

- BCFW recursion relations

- BCFW shifts and the validity
- Inductive proof of the Parke-Taylor formula
- The split-helicity NMHV amplitudes

- Validity of the recursion relations

Before proceed, I would like to remind you of some useful backgrounds from the last two talks, which will be used today.

1. Useful spin-helicity formula

$$\not{p} = -\big(|p\rangle[p| + |p]\langle p|\big) \tag{1}$$

$$\langle pq\rangle = -\langle qp\rangle, \quad [pq] = -[qp], \quad \langle pq\rangle^* = [qp] \tag{2}$$

$$\langle pq\rangle[pq] = (p+q)^2 = -s_{pq}, \quad \text{where } s_{pq} \text{ is the Mandelstam variable.} \tag{3}$$

$$\langle 1|\gamma^\mu|2]\langle 3|\gamma_\mu|4] = 2\langle 13\rangle[24] \tag{4}$$

Momentum conservation:

$$\sum_{i=1}^{n}\langle qi\rangle[ik] = 0 \tag{5}$$

Shouten identity:

$$\langle ri\rangle\langle jk\rangle + \langle rj\rangle\langle ki\rangle + \langle rk\rangle\langle ij\rangle = 0 \tag{6}$$

2. The Parke-Taylor formular

$$A_n[1^+\cdots i^-\cdots j^-\cdots n^+] = \frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}. \tag{7}$$

3. The Maximally Helicity Violating (MHV) amplitudes $A_n[1^+\cdots i^-\cdots j^-\cdots n^+]$.

1 Complex shifts and the general recursion relation

Consider an **on-shell** n-point amplitude A_n with n external **massless** momenta p_i^μ for $i = 1, 2, \cdots, n$. To use the analytic properties of A_n, we need firstly extend it to complex plane, by defining complex shifted momenta.

Introduce n complex valued vectors r_i^μ, satisfying the following conditions

$$\begin{aligned} &1)\ \sum_i r_i^\mu = 0, \\ &2)\ r_i \cdot r_j = 0, \quad r_i^2 = 0, \\ &3)\ p_i \cdot r_i = 0 \text{ (no sum)}. \end{aligned} \tag{8}$$

And using these complex-valued vectors to define the complex-valued shifted momenta

$$\hat{p}_i^\mu = p_i^\mu + z r_i^\mu, \tag{9}$$

where z is a complex variable, say $z \in \mathbb{C}$.

The constraints in eqs. (8) actually insured that the conservation and onshell condition of the shifted momenta, say

$$\begin{aligned} &1)\ \sum_i \hat{p}_i^\mu = 0, \\ &2)\ \hat{p}_i^2 = 0. \end{aligned} \tag{10}$$

Based on these two properties, we can safely write the n-point amplitude in case of the n shifted external momenta, which results in a complex valued holomorphic function, say the shifted amplitude:

$$\hat{A}_n(z) \equiv A_n(\hat{p}_1^\mu, \cdots, \hat{p}_n^\mu). \tag{11}$$

Now let's have a look at the internal propagators. The momentum of an internal propagator can be written as the summation of a nontrivial subset of external momenta:

$$P_I^\mu = \sum_{i \in I} p_i^\mu, \tag{12}$$

where I indicates the non-trivial subset. By non-trivial, we mean firstly, there are at least 2 and at most $n-2$ components, and next $P_I^2 \neq 0$. The first constraint insures that P_I^μ stands for a internal momentum rather than an external one, and the second one will show its significance later. Similarly, we have the associated shifted internal momentum $\hat{P}_i^\mu$. Let's introduce

$$R_I^\mu = \sum_{i \in I} r_i^\mu \quad \text{and} \quad z_I = -\frac{P_I^2}{2 P_I \cdot R_I}. \tag{13}$$

related to the same subset I. Using these two definitions, we find that

$$\hat{P}_I^2 = -\frac{P_I^2}{z_I}(z - z_I). \tag{14}$$

Now we can see that the complex-valued propagator $1/\hat{P}_I^2$ has a simple pole at $z = z_I$ on the complex plane of z. And the constraint $P_I^2 \neq 0$ ensures that $z \neq 0$ and that there isn't another pole at the origin as well.

At **tree level**, $\hat{A}_n(z)$ does not have brach-cuts. And all the poles of $\hat{A}_n(z)$ are given by the simple poles of propagators, say z_I. Moreover, for generic momenta, a specific propagator $1/\hat{P}_I$ can only show once in the amplitude, which means all the poles of $\hat{A}_n(z)$ lie at **different** positions z_I **away from the origin**.

Now look at the complex function $\frac{\hat{A}_n(z)}{z}$, which has three kinds of poles

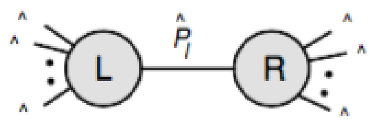
$$\begin{aligned} &1)\ z = 0, \quad \text{with the residue } A_n, \\ &2)\ z = z_I, \quad \text{with the residue } \mathrm{Res}_{z=z_I} \frac{\hat{A}_n(z)}{z}, \\ &3)\ z = \infty, \quad \text{with the residue } B_n. \end{aligned} \tag{15}$$

Now according to the Cauthy's theorem

$$A_n = -\sum_{z_I} \mathrm{Res}_{z=z_I} \frac{\hat{A}_n(z)}{z} + B_n. \tag{16}$$

Look at the Feynman diagram of $\hat{A}_n$, we can see

$$\mathrm{Res}_{z=z_I} \frac{\hat{A}_n(z)}{z} = -\hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I) = \tag{17}$$



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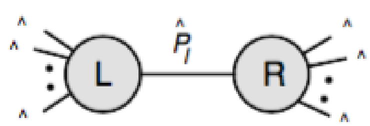
where $\hat{A}_L(z_I)$ and $\hat{A}_R(z_I)$ stand for the shifted amplitudes at the left side and the right side, evaluated at $z = z_I$. These two **subamplitudes** have less than n external legs, as long as P_I^μ is an internal momentum.

As for the residue B_n at infinity, it actually defines the validity of the recursion relation. If $B_n = 0$, we say that the recursion relation showing next is valid. The value of B_n is related to both the specific theory and the specific way to define the shift.

Now, let's assume $B_n = 0$. And the n-point amplitude A_n can be written as

$$A_n = \sum_{\text{diagrams } I} \hat{A}_L(z_I) \frac{1}{P_I^2} \hat{A}_R(z_I) = \sum_{\text{diagrams } I} \tag{18}$$



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where the summation is over both all possible diagrams giving non-trivial subset I and all possible particle states associated with that internal line. Now we successfully write the n-point amplitude in case of lower-point amplitudes, say subamplitudes $\hat{A}_L(z_I)$ and $\hat{A}_R(z_I)$, which is the most general formula of the recursion relation at tree level.

2 BCFW recursion relations

2.1 BCFW shifts and the validity

The general recursion relation (18) can not be useful unless defining a specific choice of the shift, or say the shift vectors r_i^μ. One powerful choice is called the BCFW shift.

In BCFW shift, only two external lines among the n legs, labeled by i and j, are shifted. In 4D spacetime, it's convenient to define the shift in case of the spinor helicity formula:

$$|\hat{i}] = |i] + z|j], \quad |\hat{j}] = |j], \quad |\hat{i}\rangle = |i\rangle, \quad |\hat{j}\rangle = |j\rangle - z|i\rangle, \tag{19}$$

and all the other spinors are unshifted. We label such a shift as $[i, j\rangle$. **Equivalently**, the external momenta are shifted as $\hat{p}_i^\mu = p_i^\mu - \frac{1}{2}\langle i|\gamma^\mu|j]$, $\hat{p}_j^\mu = p_j^\mu + \frac{1}{2}\langle i|\gamma^\mu|j]$ and all the other momenta are unshifted. We can check quickly that such a shift satisfies all the constraints in eqs. (8).

The recursion relation obtained based on such a shift is called the BCFW recursion relation.

Before proceeding to some real examples utilizing the BCFW recursion relation, we still need to justify its validity, or say proving that the residue B_n of the function $\hat{A}_n(z)/z$ at the infinity vanishes. To prove that, one usual strategy is to prove that

$$\hat{A}_n(z) \to 0, \quad \text{for} \quad z \to \infty, \tag{20}$$

under the shift. In pure Yang-Mills theory, the behavior of $\hat{A}_n(z)$ at large z under a BCFW shift of adjacent lines i and j of helicity as indicated is [2]

$$\begin{array}{ccccc} [i,j\rangle & [-,-\rangle & [-,+\rangle & [+,+\rangle & [+,-\rangle \\ \hat{A}_n(z) & \frac{1}{z} & \frac{1}{z} & \frac{1}{z} & z^3. \end{array} \tag{21}$$

We can see that the adjacent shift $[i, j\rangle$ is valid for the helicity of structure of $[-,-\rangle$, $[-,+\rangle$ and $[+,+\rangle$ in Yang-Mills theory,

2.2 Inductive proof of the Parke-Taylor formula

Now we use the BCFW recursion relations to **inductively** prove the Parke-Taylor formula :

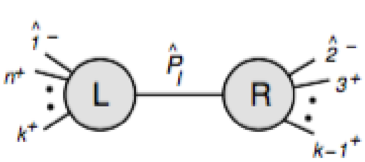
$$A_n[1^-2^-3^+\cdots n^+] = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}, \tag{22}$$

which is the adjacent case of the more general formula eq. (7).

Let's start the inductive proof with the $n = 3$ case. The $A_3[1^-2^-3^+]$ amplitude was calculated in Section 2.5 of the book, which matches the Parke-Taylor formula.

Now let's calculate the amplitude $A_n[1^-2^-3^+\cdots n^+]$ with the assumption that the formula holds for $(n-1)$ case.

Use the $[1, 2\rangle$ shifts, according to the recursion relation,

$$A_n[1^-2^-3^+\cdots n^+] = \sum_{k-4}^{n}$$


$$= \sum_{k-4}^{n}\sum_{h_I=\pm} \hat{A}_{n-k+3}[\hat{1}^-, \hat{P}_I^{h_I}, k^+, \cdots, n^+]\frac{1}{P_I^2}\hat{A}_{k-1}[\hat{P}_I^{-h_I}, \hat{2}^-, 3^+, \cdots, (k-1)^+]. \tag{23}$$

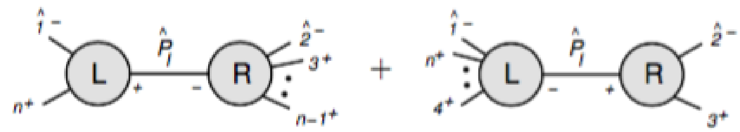
Notice for the all possible diagrams:

1. The color-ordering should be always preserved in the two subamplitudes;

2. Lines 1 and 2 should always belong to different subamplitudes. Otherwise, the internal line $\hat{P}_I$ won't be shifted (i.e., no pole contribution to the function);
3. There is also a summation over the helicities h_I of the internal line $\hat{P}_I$, which are opposite for the two subamplitudes.

Using the fact that amplitudes having the helicity structure $A_n[+ + \cdots - \cdots +]$ vanish, there are only two non-vanishing terms in eq. (23):

$$A_n[1^- 2^- 3^+ \cdots n^+] = \sum_{k-4}^{n}$$



$$\begin{aligned} &= \hat{A}_3[\hat{1}^-, -\hat{P}^+_{1n}, n^+]\frac{1}{P^2_{1n}}\hat{A}_{n-1}[\hat{P}^-_{1n}, \hat{2}^-, 3^+, \cdots, (n-1)^+] \\ &\quad + \hat{A}_{n-1}[\hat{1}^-, \hat{P}^-_{23}, 4^+, \cdots, n^+]\frac{1}{P^2_{23}}\hat{A}_3[-\hat{P}^+_{23}, \hat{2}^-, 3^+], \end{aligned} \tag{24}$$

where $P_{ij} = p_i + p_j$.

Now we can further send the $1n$ channel to be zero using the **three particle special kinematics**. According to the Parke-Taylor formula in $n = 3$ case,

$$\hat{A}_3[\hat{1}^-, -\hat{P}^+_{1n}, n^+] = \frac{[\hat{P}_{1n} n]^3}{[n\hat{1}][\hat{1}\hat{P}_{1n}]}, \tag{25}$$

where we used the following convention for analytic continuation: $| - p\rangle = -|p\rangle$, $| - p] = +|p]$. Using the on-shell condition of $\hat{P}_{1n}$:

$$\hat{P}^2_{1n} = -s_{\hat{1}n} = \langle \hat{1}n\rangle[\hat{1}n] = \langle 1n\rangle[\hat{1}n] = 0. \tag{26}$$

For generic momenta, $\langle 1n\rangle \neq 0$. So $[\hat{1}n] = 0$. Also, as

$$|\hat{P}_{1n}\rangle[\hat{P}_{1n} n] = -\hat{P}_{1n}|n] = -(\hat{p}_1 + p_n)|n] = |1\rangle[\hat{1}n] = 0, \tag{27}$$

$[\hat{P}_{1n} n] = 0$. Similarly, one can show that $[\hat{1}\hat{P}_{1n}] = 0$. Now, all spinor products in (25) vanish; with the 3 powers in the numerator versus the two in the denominator, we conclude that special 3-point kinematics force $\hat{A}_3[\hat{1}^-, -\hat{P}^+_{1n}, n^+] = 0$.

However, the special 3-point kinematics won't send the amplitude $\hat{A}_3[-\hat{P}^+_{23}, \hat{2}^-, 3^+]$. Rather than the square spinor being shifted, line 2 has its angle spinor shifted, which gives $\langle \hat{2}3\rangle = 0$, which doesn't contribute to the 3-point amplitude.

Now there is only one term left. Using the Parke-Taylor formula for the $n - 1$ case, we have

$$\begin{aligned} A_n[1^- 2^- 3^+ \cdots n^+] &= \hat{A}_{n-1}[\hat{1}^-, \hat{P}^-_{23}, 4^+, \cdots, n^+]\frac{1}{P^2_{23}}\hat{A}_3[-\hat{P}^+_{23}, \hat{2}^-, 3^+] \\ &= \frac{\langle \hat{1}\hat{P}_{23}\rangle^4}{\langle \hat{1}\hat{P}_{23}\rangle\langle \hat{P}_{23}4\rangle\langle 45\rangle \cdots \langle n\hat{1}\rangle} \times \frac{1}{\langle 23\rangle[23]} \times \frac{[3\hat{P}_{23}]^3}{[\hat{P}_{23}\hat{2}][\hat{2}3]} \\ &= \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle \cdots \langle n1\rangle}, \end{aligned} \tag{28}$$

where in the last step we have used the relations:

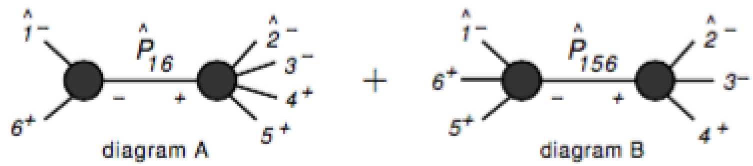
$$\langle \hat{1}\hat{P}_{23}\rangle[3\hat{P}_{23}] = -\langle 12\rangle[23], \quad \langle \hat{P}_{23}4\rangle[\hat{P}_{23}\hat{2}] = -\langle 34\rangle[23]. \tag{29}$$

Now we have completed the inductive proof of the formula.

2.3 The splitting helicity NMHV amplitude

Now let's give another example, which is the computation of the splitting helicity NMHV amplitude $A_6[1^-2^-3^-4^+5^+6^+]$. There will be many interesting problems emerging in the deviation and results.

Use the $[1,2\rangle$ shift, according to the recursion relation, there are only two non-trivial diagrams in the summation:

$A_6[1^-2^-3^-4^+5^+6^+] =$  (30)

$$= \frac{\langle 3|1+2|6]^3}{P_{126}^2[21][16]\langle 34\rangle\langle 45\rangle\langle 5|1+6|2]} + \frac{\langle 1|5+6|4]^3}{P_{156}^2[23][34]\langle 56\rangle\langle 61\rangle\langle 5|1+6|2]}.$$

Notice that the possible 23-channel, which is proportional to the three-particle amplitude $A_3[-\hat{P}_{23}^+, \hat{2}^-, 3^-] \propto \langle\hat{2}3\rangle$, vanishes due to the 3-particle special kinematics sending $\langle\hat{2}3\rangle = 0$.

Here are some interesting aspects we can look at:

*** The 3-particle pole**

There is a 3-particle pole $P_{165}^2 = 0$ showing in the diagram B, which has a factor of $1/P_{165}$. By inspection of the ordering of the external states in $A_6[1^-2^-3^-4^+5^+6^+]$, the pole $P_{165}^2 = 0$ is equivalent to the pole $P_{345}^2 = P_{126}^2 = 0$. However, there is no where to find such a pole from both diagrams directly. Actually, such a pole is hidden in the diagram A, as is shown in the second line of eq. (30).

The diagram A has a factor of $\langle\hat{2}\hat{P}_{16}\rangle$ in the denominator of the subamplitude. Notice that

$$\langle\hat{2}\hat{P}_{16}\rangle[\hat{P}_{16}3] = \langle 21\rangle[\hat{1}3] + \langle\hat{2}6\rangle\langle 63\rangle. \tag{31}$$

As $z_{16} = \langle 16\rangle/\langle 26\rangle$, we can rewrite $\langle\hat{2}6\rangle = (\langle 16\rangle[16] + \langle 26\rangle[26])/\langle 26\rangle$ and $[\hat{1}3] = [12][36]/[26]$, which further gives

$$\langle\hat{2}\hat{P}_{16}\rangle[\hat{P}_{16}3] = -\frac{[36]}{[26]}P_{126}^2. \tag{32}$$

Now we see how the equivalent 3-particle pole $P_{126}^2 = 0$ shows up in the BCFW results.

*** The $[2,1\rangle$ shift**

If we use the $[2,1\rangle$ shift, there are actually three non-trivial diagrams in the recursion relation. There's now a problem how to show the results calculated from the two shifts ($[2,1\rangle$ with three diagrams and $[1,2\rangle$ with two diagrams) are the same. Using the Schouten identities and momentum conservation, one can prove this with brutal force. Another way is the numerical check. However, the most elegant way is to use the residue theorem, also where such a discrepancy originates from [3], and a description of amplitudes in the Grassmannian, discussed later in the book.

*** Spurious poles**

Color-ordered tree amplitudes can have physical poles only when the momenta of adjacent external lines go **collinear**. For example, the 3-particle poles mentioned above. But notice that the factor $\langle 5|1+6|2]$ in the denominator for each BCFW diagram in eq. (30), will go zero in the collinear limit. But this does not correspond to a physical pole of the scattering amplitude: it is a **spurious pole**. The residue of this unphysical pole better be zero and it is: the spurious pole cancels in the sum of the two BCFW diagrams in (30). It is typical that BCFW packs the information of the amplitudes into compact expressions, but the cost is the appearance of spurious poles; this means that in the BCFW representation the locality of the underlying field theory is not manifest. Elimination of spurious poles in the representations of amplitudes leads to interesting results [4] that discussed later in the book.

3 Validity of the recursion relations

As we have just seen, with the on-shell BCFW recursion relations, we can construct all higher-point gluon tree amplitudes from the input of just the 3-point gluon amplitudes. That is a lot of information obtained from very little input! This point is actually insured by symmetries of the theory. For example, the gauge symmetry in the Yang-Mills theory. We'll illustrate this with three examples here.

3.1 Scalar QED

In scalar QED, the interaction between photons and the scalars is encoded by

$$\mathcal{L} \supset -|D\phi|^2 = |\partial\phi|^2 + ieA^\mu\big[(\partial_\mu\phi^*)\phi - \phi^*\partial_\mu\phi\big] - e^2 A_\mu A^\mu \phi^*\phi. \tag{33}$$

The 4-point amplitude $A_4(\phi\phi^*\gamma\gamma)$ can be constructed via the BCFW recursion relations, from the 3-point amplitude $A_3(\phi\phi^*\gamma)$. This can be understood that regarding the Lagrangian, only the information in the 3-point vertices is needed, and the role of the $A_\mu A^\mu\phi^*\phi$ vertex is just to ensure off-shell gauge invariance of the Lagrangian.

Now let's look at another 4-point amplitude, say $A_4(\phi\phi^*\phi\phi^*)$. Using a $[1,3\rangle$ shift, it can be shown that the BCFW recursion relations give

$$A_4^{\text{BCFW}}(\phi\phi^*\phi\phi^*) = \tilde{e}^2\frac{\langle 13\rangle^2\langle 24\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{34}$$

However, using Feynman rules, the result is

$$A_4^{\text{Feynman}}(\phi\phi^*\phi\phi^*) = \tilde{e}^2\Big(1 + \frac{\langle 13\rangle^2\langle 24\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\Big). \tag{35}$$

We see there's now a discrepancy between. Dose this mean the BCFW recursion relations fail for scalar QED?

Actually, the the BCFW recursion relation result, say eq. (34), is a special case of the general family of scalar-QED models. More generally, we should also include a term $\lambda|\phi|^4$ and consider the scalar-QED action from

$$\mathcal{L} \supset -|D\phi|^2 - \frac{1}{4}\lambda|\phi|^4 = |\partial\phi|^2 + ieA^\mu\big[(\partial_\mu\phi^*)\phi - \phi^*\partial_\mu\phi\big] - e^2 A_\mu A^\mu\phi^*\phi - \frac{1}{4}\lambda|\phi|^4, \tag{36}$$

which gives the amplitude $A_4(\phi\phi^*\phi\phi^*)$ as

$$A_4(\phi\phi^*\phi\phi^*) = -\lambda + \tilde{e}^2\Big(1 + \frac{\langle 13\rangle^2\langle 24\rangle^2}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\Big). \tag{37}$$

Now it is clear that the BCFW result eq. (34) is the special case of eq. (37) when $\lambda = \tilde{e}^2$. It is actually exactly such a constraint that justifies the validity of the recursion relations, in the sense of the boundary term B_n. Under the $[1,3\rangle$-shift, which we used to derive the result in (34), the boundary term has a value of $-\lambda + \tilde{e}^2$. The special choice $\lambda = \tilde{e}^2$ eliminates the boundary term, which is required by the validity of the recursion relations.

The lesson is that for general, there is no way in which the 3-point interactions can know the contents of $\lambda|\phi|^4$: it provides independent gauge-invariant information. However, given proper symmetries, we do can determine the information in $\lambda|\phi|^4$ in terms of the 3-field terms. In fact, the expression (34) actually occurs for 4-scalar amplitudes in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ SYM theory, and in those cases the coupling of the 4-scalar contact term is fixed by the Yang-Mills coupling by **supersymmetry**.

3.2 Yang-Mills theory

Now it is easy to understand what happens in the Yang-Mills theory, where we can determine higher order gluon scattering amplitudes from 3-point amplitudes, using the BCFW recursion relations. The reason it the **gauge symmetry**, since the A^4 term in the Lagrangian is fully determined from $A^2\partial A$ by the requirement of the off-shell gauge invariance of the Lagrangian, it contains no new on-shell information.

3.3 Scalar theory $\lambda\phi^4$

The last example I'd like to talk about is the scalar theory $\lambda\phi^4$, with another interesting feature. Suppose we just consider $\lambda\phi^4$-theory with no other interactions. The recursion relations actually start with the 4-point amplitudes. To justify the recursion relations, we need to find the boundary term B_n, or say the $\mathcal{O}(z^0)$-behavior for large z of the amplitudes, and send it to zero. Inspection of the Feynman diagrams reveals that $\mathcal{O}(z^0)$-contributions are exactly the diagrams in which the two shifted lines belong to the same vertex. And the sum of such diagrams equals the boundary term B_n. One can in this case of $\lambda\phi^4$-theory reconstruct B_n **recursively**. Thus we can use recursion relations to compute high-order amplitudes A_n with only input from the 4-point amplitude A_4.

References

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[2] N. Arkani-Hamed and J. Kaplan, JHEP 0804, 076 (2008) [arXiv:0801.2385 [hep-th]];

[3] N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, JHEP 1003, 020 (2010) [arXiv:0907.5418 [hep-th]];

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