Renormalization Running of Entanglement Entropy

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1. Example: Massive scalar EE in (1+1) D [Ref: Calabrese & Cardy 0905.4013, Section 5]
2. General R.N. Running of EE in (1+1) D [Ref: Casini & Huerta 1202.5650]
3. General R.N. Running of EE in (2+1) D [same Ref as 2]

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1. Consider a theory with one mass scale m breaking the conf. inv.. (e.g. a massive scalar field)

On the n-sheeted Riemann surface $\mathcal{R}^n$, by examing rotational and translational properties we can write $\langle T_{zz}(z,\bar z)\rangle_{\mathcal{R}^n} = \frac{F_n(|z|^2)}{z^2}$, $\langle T^\mu{}_\mu(z,\bar z)\rangle_{\mathcal{R}^n} = \langle 4T_{z\bar z}(z,\bar z)\rangle_{\mathcal{R}^n} = \frac{G_n(|z|^2)}{|z|^2}$, $\langle T_{\bar z\bar z}(z,\bar z)\rangle_{\mathcal{R}^n} = \frac{F_n(|z|^2)}{\bar z^2}$.

In particular, for $n=1$, $F_1=0$, $G_1(|z|^2) = g_1|z|^2$ for some constant g_1. Write $\tilde G_n = G_n - G_1$.

$\nabla^\mu T_{\mu\nu} = 0 \Rightarrow |z|^2(F_n' + \frac{1}{4}\tilde G_n') = \frac{1}{4}\tilde G_n$.

For $|z|^2$ running from 0 to ∞, the theory runs from UV to IR fixed points. Near fixed points, F_n and G_n should be close to their CFT values: $F_n \approx \frac{C_{UV} \text{ or } C_{IR}}{24}(1-\frac{1}{n^2})$, $\tilde G_n \approx 0$.

Thus, ~~solving~~ integrating the diff. eq. above $\Rightarrow \int_0^\infty \frac{\tilde G_n(|z|^2)}{|z|^2} d|z|^2 = \frac{C_{UV}-C_{IR}}{24}(1-\frac{1}{n^2})$

The LHS is $\frac{1}{n}\int(\langle T^\mu{}_\mu\rangle_{\mathcal{R}^n} - \langle T^\mu{}_\mu\rangle_{\mathcal{R}^1})\, d|z|^2 = \frac{1}{n\pi}\int(\langle T^\mu{}_\mu\rangle_{\mathcal{R}^n} - \langle T^\mu{}_\mu\rangle_{\mathcal{R}^1})\, d^2z$ by rotational symm.

To relate this to R.N., note that when there is only 1 dimensionful parameter, $\int\langle T^\mu{}_\mu\rangle_{\mathcal{R}^n} d^2z = -2\pi\frac{\partial \ln Z_{\mathcal{R}^n}}{\partial \ln m}$.

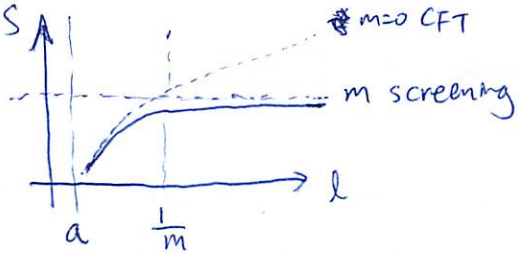
Thus integrating m we get $\frac{Z_{\mathcal{R}^n}}{Z_{\mathcal{R}^1}^n} = C_n(m\tilde a)^{\frac{C_{UV}-C_{IR}}{12}(n-\frac{1}{n})}$

where C_n is independent of m (but may depend on e.g. length of slit l), $C_1=1$. a is cutoff scale, $ma \ll 1$. $\tilde a$ depends on a and l, $\tilde a$ has the same dimension as a.

$S = -\lim_{n\to 1}\frac{\partial}{\partial n}\frac{Z_{\mathcal{R}^n}}{Z_{\mathcal{R}^1}^n} = \frac{C_{UV}-C_{IR}}{6}\ln\frac{1}{ma} + $ (other m, l dependences)

The effect of length l of slit is in the latter term. The first term should dominate for $l \gg \frac{1}{m}$, since the correlation length $\sim\frac{1}{m}$ would "screen out" l. While, at $l \ll \frac{1}{m}$, the mass is negligiable, the second term should approach the CFT result $\frac{C_{UV}}{3}\ln\frac{l}{a} + \text{const}$.

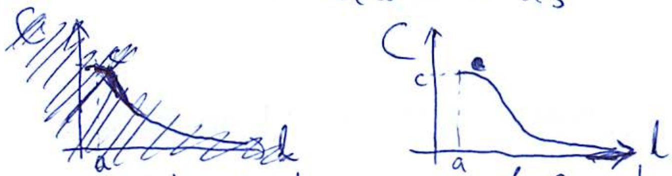
Thus $\underset{\ell \gg a}{S(\ell)}$ looks like



Define functions $C(\ell) \equiv 3\ell S'(\ell)$, $C_0(\ell) \equiv S(\ell) - \ell \ln\left(\frac{\ell}{a}\right) S'(\ell)$.

For CFT, where $S(\ell) = \frac{c}{3}\ln\left(\frac{\ell}{a}\right) + \text{const.}$, $C(\ell) = c$, $C_0(\ell) = \text{const.}$

Now, with m, $C(\ell)$ runs as



And though the running of C_0 is hard to see pictorially, we can see

$$C_0'(\ell) = S' - S' - \ln\left(\tfrac{\ell}{a}\right) S' - \ell \ln\left(\tfrac{\ell}{a}\right) S'' = -\ln\left(\tfrac{\ell}{a}\right) C'(\ell) > 0 \ \forall \ell.$$

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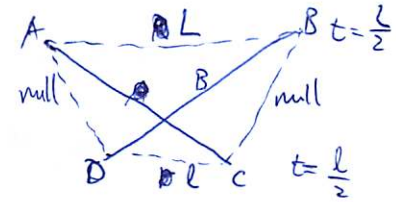
2. Can the running be generalized? ~~How~~ Is the decreasing of $C(\ell)$ related to the c-theorem?



Since $S(\ell)$ only depends on ℓ, consider in space time

By simple Mink. argument, the space time lengths of AC and BD are $\sqrt{\ell L}$.

Strong subadditivity tells $S(AD \cup DC) + S(BC \cup CD) \geq S(CD) + S(AD \cup CD \cup BC)$

Unitarity tells $S(AD \cup DC) = S(AC)$, $S(BC \cup CD) = S(BD)$, $S(AD \cup CD \cup BC) = S(AB)$

But S only depends on length for any QFT. So we get

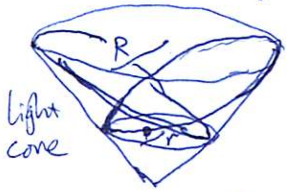
$2S(\sqrt{\ell L}) \geq S(L) + S(\ell)$

Let $L = \ell + \varepsilon \Rightarrow C'(\ell) = \ell S''(\ell) + S'(\ell) \leq 0$.

So this is an EE version of c-theorem.

$C_0' = -\ln\left(\frac{\ell}{a}\right) C'(\ell) \geq 0$ is also obvious.

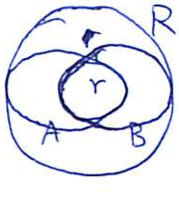
3. Can the argument be generalized to higher dimensions? Consider (2+1)D.

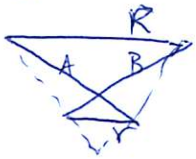


By analogy we can consider this picture. But the problem is the intersection of A, B, $\underbrace{\text{is}}_{\text{(projected back to } t=r)}$  which is not ~~a circle~~. not a disk.

top view

side view





So instead to two disks A, B, we need N of them, $N \to \infty$. ~~so that~~

More precisely, the strong subaddtivity for more regions is

$$\sum_i S(X_i) \geq S\Big(\bigcup_i X_i\Big) + S\Big(\bigcup_{i<j}(X_i\cap X_j)\Big) + S\Big(\bigcup_{i<j<k}(X_i\cap X_j\cap X_k)\Big) + \cdots + S\Big(\bigcap_i X_i\Big)$$

For X_i a boosted disk like A above but X_{i+1} and X_i boosted differ by an angle $\frac{2\pi}{N}$, the above is

$$N S(\sqrt{Rr}) \geq \sum_{n=0}^{N} S(r_n)$$ where r_n is the "radius" of the "approximate disk" formed by the ~~intersection of $X_1\cap X_{1+n}\cap X_{1+2n}\cdots X_N$~~ union of $X_i\cap X_{i+n}$ (smallest for arbitrary intersection of N-n disks). $r_0=r$, $r_N=R$.

By some geometrical arguments one can work out

$$r_n = \frac{2rR}{(R+r)+(R-r)\cos\left(\frac{\pi n}{N}\right)}$$

Taking the limit $N\to\infty$, we have

$$S(\sqrt{Rr}) \geq \frac{1}{\pi}\int_0^{\pi} d\theta\, S\left(\frac{2rR}{R+r-(R-r)\cos\theta}\right) = \frac{1}{\pi}\int_r^R ds\,\frac{\sqrt{Rr}}{s\sqrt{(R-s)(s-r)}}\,S(s)$$

where the last step is a change of variable.

Expanding $R=r+\varepsilon$ and after some calculation one finds

$S'' \leq 0$.

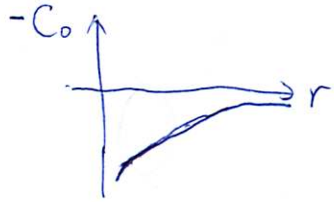
In particular, for CFT we know $S(r) = C_1 r + C_0$

Now for general QFT define $C_1(r) \equiv S'(r)$ and $C_0(r) \equiv S(r) - rS'(r)$.

Then $C_1' \leq 0$, $C_0' = S' - S' - rS'' \geq 0$

(For (d+1) dim $\boxed{\text{spheres}}$, the method provides $rS''(r) - (d-2)S' \leq 0$)

There has been discussion whether $-C_0(r)$ can play the role of C-function.
But it is found that $-C_0(r)$ runs as



for a massive scalar.
Near the UV fixed point $-C_0'$ does not approach 0,
so it's not likely a "C-function".