Boson Particle/Vortex Duality

Ian MacCormack

October 27, 2016

We express a system of lattice bosons in terms of vortex excitations by introducing an additional gauge field. We then take the continuum limit to find a duality between ϕ^4 theory of a complex scalar and an abelian Higgs model describing the vortices. This allows us to map the superfluid phase of the bosons onto the Coulomb phase of the Higgs model, and the Mott insulating phase of the bosons onto a vortex condensate.

1 Lattice Derivation

We start with a system of free bosons hopping on a 2D lattice

$$H = -t \sum_{<ij>} (b_i^\dagger b_j + b_j^\dagger b_i)$$

Representing the operators instead by their phase angle, we make the following substitution

$$b_i = e^{i\phi_i} \rightarrow \mathcal{S} = -K \sum_{\mu,i} cos(\Delta_\mu \phi_i)$$

To make the action a bit easier to deal with, we can represent each Boltzmann factor in the partition function by a sum of gaussians, which has the same behavior as the original cosine action near integer values of $\Delta_\mu \phi_i$.

$$e^{-K(1-cos(\Delta_\mu \phi_i))} = \sum_{m_{\mu,i}} e^{-K/2(\Delta_\mu \phi_i - 2\pi m_{\mu,i})^2}$$

Where this is a good approximation for large K, but retains the essential periodicity of the original function for any value of K. Introducing this sum has the effect of gauging the global $U(1)$ symmetry. We can now make integer gauge transformations to ϕ, which can be absorbed by a complementary transformation of the newly introduced integer gauge field, $m_{\mu i}$. Intuitively, this gauge field can be thought of as containing the singular, vortex-like behavior of the boson phase angle. This interpretation will be useful later when defining the vorticity.

Our action is now:

$$\mathcal{S} = -K/2 \sum_{m_{\mu,i}} (\Delta_\mu \phi_i - 2\pi m_{\mu,i})^2$$

Introducing an auxiliary field, $p_{\mu,i}$ we get:

$$\mathcal{S} = \frac{1}{2K} \sum_{\mu,i} p_{\mu,i}^2 + i p_{\mu,i}(\Delta_\mu \phi_i - 2\pi m_{\mu,i})\phi_i$$

We can then integrate by parts:

$$\mathcal{S} = \frac{1}{2K} \sum_{\mu,i} p_{\mu,i}^2 - i\phi_i \Delta_\mu p_{\mu,i} + i p_{\mu,i} m_{\mu,i}$$

We will soon see that the auxiliary variable has a very clear interpretation as boson current. Performing the functional integral over ϕ_i yields a delta functional that enforces the following condition on the auxiliary field, $p_{\mu i}$:

$$\rightarrow \Delta_\mu p_{\mu i} = 0$$

We can meet this condition by setting $p_{\mu i}$ equal to a pure curl, so that it is exact (and thus, current-like):

$$\rightarrow p_{\mu,i} = \epsilon_{\mu\nu\lambda}\Delta_\nu a_{\lambda i} = \sum_{\nu\lambda} \epsilon_{\mu\nu\lambda}(p_{i-\lambda,\lambda} - p_{i-\nu-\lambda,\lambda})$$

Using this, and rearranging the sum in the second term, our action now takes the following form:

$$\mathcal{S} = \frac{1}{2K}\sum_{\mu,i}(\epsilon_{\mu\nu\lambda}\Delta_\nu a_{\lambda,i})^2 - ia_{\mu,i}\epsilon^{\mu\nu\lambda}\Delta_\nu m_{\lambda,i}$$

where the discrete curl of m_μ is defined the same way as it was for p_μ. We have now captured the dynamics of the original boson field in a dynamical gauge field. But where are the vortices, you might ask? We can identify

$$M_{\mu,i} = \epsilon_{\mu\nu\lambda}\Delta_\nu m_{\lambda,i}$$

as the vorticity current, given our previous interpretation of m as related to the singular behavior of the original boson phase field. If we introduce a new angular scalar field, θ_i, that may have singular behavior so that $m_{\mu,i} = \Delta_\mu$, we can get the same current by minimal coupling of θ_i to the emergent gauge field.

$$\mathcal{S}_{\mathcal{QED}} = \frac{1}{2K}\sum_{\mu,i}(\epsilon_{\mu\nu\lambda}\Delta_\nu a_{\lambda i})^2 - \sum_j(\Delta_\mu\theta_j - a_{\mu j})^2$$

$$\mathcal{Z} = \prod_i \int \mathcal{D}\theta_i \prod_{\mu,i}\mathcal{D}a_{\mu j}e^{-\mathcal{S}}$$

The new scalar field now has the interpretation of vortex, and action now has the form of scalar QED. Let us ensure that our interpretation of a_μ as related to the boson dynamics is justified by inserting an external gauge field in the original action and seeing what it couples to:

$$\mathcal{S}_{\mathcal{QED}} = -K\sum_i cos(\Delta_\mu\phi_i - A_{\mu i}) \rightarrow \mathcal{S}_{\mathcal{QED}} + \sum_{\mu,i}\frac{i}{2\pi}A_{\mu i}\epsilon_{\mu\nu\lambda}\Delta_\nu a_{\lambda i}$$

$$J_{\mu i} = \frac{i}{2\pi}\epsilon_{\mu\nu\lambda}\Delta_\nu a_{\lambda i}$$

As we can see, a_μ clearly generates the boson current that couples to the external gauge field. Overall, we have a scalar vortex field minimally coupled to a dynamical gauge field that represents the original boson particles.

2 Continuum Limit and Analysis

Near the insulator-superconductor phase transition, the original boson model looks like the following in the continuum:

$$\varphi \sim e^{i\phi} \rightarrow \mathcal{Z}_\varphi = \int \mathcal{D}\varphi e^{-S_\varphi}$$

$$S_\varphi = \int d^3x(|\partial_\mu\varphi|^2 + r|\varphi|^2 + g|\varphi|^4)$$

where r and g are undetermined coupling constants.

We can now follow a similar, albeit slightly easier, derivation to the one we did on the lattice. First, assume that ϕ is near a minimum of the potential, so that we can write it as $\sqrt{\rho}e^{i\theta}$. Then add an external gauge field, A_μ. We get the following Lagrangian:

$$\mathcal{L}_\phi = |(\partial_\mu - A_\mu)\phi|^2 = \rho(\partial_\mu\theta - iA_\mu)^2$$

We can then perform a Hubbard-Stratanovich transformation, as in the lattice model:

$$\mathcal{L} = \frac{1}{\rho}p_\mu^2 + p^\mu(\partial_\mu\theta - iA_\mu)$$

Separating $\theta = \theta_{vor} + \theta_{smooth}$ into singular vortex and smooth parts, we get:

$$\mathcal{L} = \frac{1}{\rho}p_\mu^2 + p^\mu(\partial_\mu\theta_{vor} + \partial_\mu\theta_{smooth} - iA_\mu)$$

Integrating over the smooth part of θ gives the familiar condition:

$$\partial_\mu p^\mu = 0 \rightarrow p^\mu = \epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda$$

So now:

$$\mathcal{L} = \frac{1}{\rho}f_{\mu\nu}f^{\mu\nu} + \epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda(\partial_\mu\theta_{vor} - A_\mu)$$

We see once again that the boson current is indeed $\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$. This term also makes it evident, after integration by parts, that the vortex current coupled to the dynamical gauge field a_μ is $J^\lambda_{vor} = \epsilon^{\lambda\mu\nu}\partial_\mu\partial_\nu\theta_{vor}$. We could also write the coupling between the dynamical gauge field and the vortices as a minimal coupling between a_μ and a complex scalar field $\Phi \sim e^{i\theta_{vor}}$. We write this theory down, ignoring the external gauge field:

$$\Phi \sim e^{i\theta_{vor}} \rightarrow \mathcal{Z}_\Phi = \int \mathcal{D}\Phi\mathcal{D}a_\mu e^{-S_\Phi}$$

$$S_\Phi = \int d^3x(|(\partial_\mu - ia_\mu)\Phi|^2 + s|\Phi|^2 + v|\Phi|^4 + \frac{1}{K}f_{\mu\nu}f^{\mu\nu})$$

Once again, no constraints have been put on the coupling constants. They can be arbitrarily large.

Let's look at the superfluid phase of the vortices, in which Φ has a nonzero VEV Φ_0, plus small fluctuations:

$$\Phi \rightarrow \Phi_0 + \delta\Phi$$

Plugging the above form of the field into the action yields an effective mass term for the emergent gauge field:

$$S_\Phi = \int d^3x ... + \Phi_0^2 a_\mu a^\mu + ...$$

This tells us that the symmetry breaking superfluid phase of the vortices (the vortex condensate) corresponds directly to the insulating phase of the original bosons. The original bosons (which act as the photons of the theory) have acquired a mass term and are gapped - a Mott insulator. i.e.:

$$s < s_{crit} \rightarrow r > r_{crit}$$

Reactivating the external gauge field to examine the insulating phase of the vortices:

$$S'_\varphi = \int d^3x(|(\partial_\mu - iA_\mu)\varphi|^2 + r|\varphi|^2 + g|\varphi|^4) \rightarrow S_\Phi = \int d^3x(|(\partial_\mu - ia_\mu)\Phi|^2 + s|\Phi|^2 + v|\Phi|^4 + \frac{1}{K}f_{\mu\nu}f^{\mu\nu} + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda)$$

Now if we let the VEV of the vortices equal zero

$$\Phi \rightarrow 0$$

we obtain a theory of gapless excitations of the emergent gauge field coupled to the external gauge field - aka the superfluid phase of the original bosons.

$$S_\Phi = \int d^3x(\frac{1}{K}f_{\mu\nu}f^{\mu\nu} + \frac{i}{2\pi}\epsilon_{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda)$$

If we instead allow Φ to be small and integrate out a_μ (still in the gapped vortex phase), we find an effective logarithmic interaction between vortices, mediated by the gapless bosons, which now play the role of the photon.