A Brief Introduction to Duality Web

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Abstract: This note is prepared for the journal club talk given on Nov. 10th, 2016, where we review some IR dualities derived in recent studies.

Contents

In previous talks we have gone through various types of dualities. Though we may have acquired a rough idea about duality, that 2 seemingly distinct theories actually describe the same physical essence, we did not really specify what it means by duality. Indeed, the definition of this term is rather vague. For examples, in classical Ising model we can consider expansions on the physical lattice and the dual lattice, which respectively correspond to expansions at opposite temperature scale. To that end, we actually try to formulate theories defined on lattices with *dual geometries*, say square $\leftrightarrow$ square and triangle $\leftrightarrow$ honeycomb and dig out the correspondence.

Then we have gone over several lattice gauge theory. Using techniques such as Poisson summation we are able to represent lattice XY model in seemingly distinct representations, for instance, the current loop expansion. Some operator correspondence were established such as one between a compact boson ϕ and a massless gauge field a_μ, $\partial_\mu \phi \leftrightarrow \epsilon_{\mu\nu\rho}\partial^\nu a^\rho$. At this level, duality refers to phrasing the same partition function in terms of different dynamical variables and thus establish the equivalence between different lattice models. In the course of breathing life to some of Hubbard-Stratonovich fields, the mapping is no longer exact owing to extra dynamical terms. This kind of duality belongs to the so-called IR dualities, which means 2 theories flow to the same one at IR fixed point. Since we will be discussing the generalization of the bosonic particle-vortex duality, in the rest of this note, by duality I will be referring to IR duality. In addition, most of the time we will turn on background fields coupling to conserved currents on both sides. Dualities then can be phrased as the equivalence of partition functions as functions of background fields.

Although an IR duality, say one between theory $\mathscr{A}$ and $\mathscr{B}$, is exact only at the critical point, we may imagine if we slightly move away from the critical point, the deformation $\delta_{\mathscr{A}}$ maps, somehow, to $\delta_{\mathscr{B}}$. Consequently, we may still compare the phase relations between perturbed theories $\mathscr{A}'$ and $\mathscr{B}'$.

Besides, although this note aims to explain those examples in Ref. [10], we borrow some arguments from Ref. [8] from sections to sections and some other related dualities proposed by other works such as Ref. [3, 5, 15], to name a few.

1 The Bricks

In this section we introduce/state the fundamental building blocks and techniques in the following derivations.
Our starting duality is one relating a free fermion to a scalar coupled to U(1) theory. In the course of the note, we denote dynamical fields using lowercase letters and background fields using capital letters. Turning on the background field A, it reads

$$\mathrm{i}\bar{\Psi}\not{D}_A\Psi \leftrightarrow |D_b\phi|^2 - |\phi|^4 + \frac{1}{4\pi}b\,\mathrm{d}b + \frac{1}{2\pi}b\,\mathrm{d}A. \tag{1.1}$$

Since it relates 1 boson theory to 1 fermion one, sometimes it is called 3 dimensional bosonization.
An operator mapping can be drawn immediately by varying both side with respect to A_μ, responding to which is the electromagnetic current J^μ.

$$\bar{\Psi}\gamma^\mu\Psi \leftrightarrow \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\mu a_\rho. \tag{1.2}$$

Recall that in 1+1 dimensions, phrasing a fermion in terms of bosons makes use of Jordan-Wigner transformation, where an extended bosonic configuration is needed so as to incorporate strong correlation between fermions. By virtue of that, we suspect the fermion Ψ maps also to some composite extended operators on the right-hand side. Let us look more closely at the case $\mathrm{d}A = 0$. Variation with respect to a_0 on the right-hand side yields

$$\rho_\phi + \frac{1}{2\pi}\mathrm{d}b = \rho_\phi + \frac{1}{2\pi}f = 0, \tag{1.3}$$

where $j^\mu_\phi = \mathrm{i}\phi^\dagger \overleftrightarrow{D}^\mu\phi$. Explicitly, the flux of emergent gauge field $\mathrm{d}b$ is proportional to the charge density of ϕ. Owing to this fact, the theory on the right-hand side sometimes is regarded as the relativistic version of flux attachment. On top of that, this relation implies that when there is a single monopole (vortex) f_M event satisfying $\int \mathrm{d}b = \int f_M = 2\pi \Rightarrow Q_\phi = -1$. Another $\phi^\dagger$ exists. Consequently, we have an operator

$$\phi^\dagger f_M \tag{1.4}$$

being neutral under $\mathrm{U}_b(1)$ but carrying charge $+1$ under $\mathrm{U}_A(1)$. We then state the correspondence

$$\Psi \leftrightarrow \phi^\dagger f_M. \tag{1.5}$$

If we turn on the mass $m\bar{\Psi}\Psi$, depending on the sign of m, or the sign relative to one of Pauli-Villar mass, the fermion theory acquires Hall conductance 0 or -1. On the other hand, this deformation also turns on a mass term $r|\phi|^2$ on the boson side. When $r < 0$, Higgs mechanism makes the IR theory trivial, corresponding to $\nu = 0$. On the other hand, when $r > 0$, ϕ is gapped.

$$\frac{1}{4\pi}b\,\mathrm{d}b + \frac{1}{2\pi}b\,\mathrm{d}A = \frac{1}{4\pi}(b+A)\,\mathrm{d}(b+A) - \frac{1}{4\pi}A\,\mathrm{d}A, \tag{1.6}$$

corresponding to $\nu = -1$. We would like to spend a paragraph to discuss how time-reversal T is implemented on the effective action (as I understand.) Classically, a single massless Dirac cone in 2+1 dimensions is time-reversal invariant. Nonetheless, quantum mechanically, there is no way to properly calculate the partition function while preserving both U(1) and time-reversal symmetry. This is the origin of *parity anomaly* [2]. For example, we can introduce a gauge invariant Pauli-Villar regulator into a theory when calculating the fermion determinant, yet the heavy mass of the Pauli-Villar field explicitly breaks time-reversal. This issue is addressed again recently in the review paper Ref. [14]. In particular, a regularization yields the result

$$\det[\not{D}_A] = |\det[\not{D}_A]|\, \mathrm{e}^{-\mathrm{i}\pi\frac{\eta}{2}}. \tag{1.7}$$

Other regularizations are realized by adding properly normalized counter terms into the Lagrangian, taking $\frac{1}{4\pi}A\,\mathrm{d}A$ for example. The parity anomaly, from this point of view, manifests from the imaginary part of effective actions. On some under time-reversal, $\eta \to -\eta$. If we want to preserve the regularization convention, equivalently a term η is added to the effective action. In flat space, $\pi\eta$ can be replaced with $\frac{1}{4\pi}\int \mathrm{d}^3x\, A\,\mathrm{d}A$ according to Atiyah-Padoti-Singer (APS) index theorem. Thus, for a single Dirac cone $\mathrm{i}\bar{\Psi}\not{D}_A\Psi \to \mathrm{i}\bar{\Psi}\not{D}_A\Psi + \frac{1}{4\pi}A\,\mathrm{d}A$,[1] being not time-reversal invariant. Another way of thing this term is that the Pauli-Villar mass is T odd, and thus the change in Lagrangian is $\frac{1}{4\pi}A\,\mathrm{d}A$.
A way to store T symmetry (of the partition function) is including bulk contribution into the theory $\frac{1}{8\pi}A\,\mathrm{d}A$, which can be thought as a 3+1 dimensional θ term taking its value on the boundary with $\theta = \pi$. Including this term, the partition function becomes real. More precisely, $|\det[\not{D}_A]|(-1)^{\mathfrak{I}}$, where $\mathfrak{I}$ is the 3+1 dimensional Dirac index [14].

2 Dualities

In this section let us derive other dualities using (1.1). This is done by gauging the original background field and turn on another background field (since in 2+1 dimensions a gauged U(1) symmetry yields another U(1) conserved current.) That is to say, we let $A \to a$ and add on both side of the duality $\frac{1}{2\pi}a\,\mathrm{d}B - \frac{1}{4\pi}B\,\mathrm{d}B$, where B is the new background field. The duality becomes

$$\mathrm{i}\bar{\Psi}\not{D}_a\Psi + \frac{1}{2\pi}a\,\mathrm{d}B - \frac{1}{4\pi}B\,\mathrm{d}B \leftrightarrow |D_b\phi|^2 - |\phi|^4 + \frac{1}{4\pi}b\,\mathrm{d}b + \frac{1}{2\pi}(b+B)\mathrm{d}a - \frac{1}{4\pi}B\,\mathrm{d}B. \tag{2.1}$$

Integration over a on the right-hand side yields functional delta function imposing $b = -B$. The following integration over b leads to a scalar at Wilson-Fisher fixed point, while the fermion side is a single fermion coupled to a QED.

$$\mathrm{i}\bar{\Psi}\not{D}_a\Psi + \frac{1}{2\pi}a\,\mathrm{d}B - \frac{1}{4\pi}B\,\mathrm{d}B \leftrightarrow |D_{-B}\phi|^2 - |\phi|^4. \tag{2.2}$$

[1] There should be also gravitational Chern-Simons term. We suppress it throughout this note since we work in flat spacetime.

2.1 Bosonic Particle-Vortex Duality

Using (2.2), we gauge $B \to b$ and turn on another background field C. The fermion side becomes

$$\mathrm{i}\bar{\Psi}\not{D}_a\Psi + \frac{1}{2\pi}a\,\mathrm{d}b - \frac{1}{4\pi}b\,\mathrm{d}b + \frac{1}{2\pi}b\,\mathrm{d}C \tag{2.3}$$

while the boson side becomes the Abelian Higgs model

$$|D_{-b}\phi|^2 - |\phi|^4 + \frac{1}{2\pi}b\,\mathrm{d}C. \tag{2.4}$$

Next we try to integrate out b, whose equation of motion is $\mathrm{d}(b-a-C)=0$. Thus we substitute $b = a + C$ into the action. Those Chern-Simons terms now read

$$\begin{aligned}&\frac{1}{2\pi}a\,\mathrm{d}(a+C) - \frac{1}{4\pi}(a+C)\,\mathrm{d}(a+C) + \frac{1}{2\pi}(a+C)\,\mathrm{d}C\\ =&\frac{1}{4\pi}a\,\mathrm{d}a + \frac{1}{4\pi}\,C\,\mathrm{d}C + \frac{1}{2\pi}a\,\mathrm{d}C.\end{aligned}$$

Next we group the fermion and $\frac{1}{4\pi}a\,\mathrm{d}a$ Lagrangian. This combination can be regarded as a single fermion in 2+1 dimensions under time reversal operation. Thus it is dual to the time reversed version of the right-hand side of (1.1), which reads

$$\mathrm{i}\bar{\Psi}\not{D}_a\Psi + \frac{1}{4\pi}a\,\mathrm{d}a \leftrightarrow |D_{-b}\tilde{\phi}|^2 - |\tilde{\phi}|^4 - \frac{1}{4\pi}b\,\mathrm{d}b - \frac{1}{2\pi}b\,\mathrm{d}a. \tag{2.5}$$

Integrating over a imposes the constraint $b = C$ and we end up with the duality

$$|D_{-C}\tilde{\phi}|^2 - |\tilde{\phi}|^4 \leftrightarrow |D_{-b}\phi|^2 - |\phi|^4 + \frac{1}{2\pi}b\,\mathrm{d}C. \tag{2.6}$$

This is the bosonic particle-vortex duality mentioned introduced in previous talks.

2.2 Fermionic Particle-Vortex Duality

Next we discuss the fermion particle-vortex duality. This can be derived by adding $-\frac{1}{4\pi}B\mathrm{d}B$ to (2.2), making it dynamical $B \to b$ and coupling it to another new background field A. Replacing Ψ with χ, the fermion side becomes

$$\mathrm{i}\bar{\chi}\not{D}_a\chi + \frac{1}{2\pi}a\,\mathrm{d}b - \frac{2}{4\pi}b\,\mathrm{d}b + \frac{1}{2\pi}b\,\mathrm{d}A. \tag{2.7}$$

The boson side then reads

$$|D_{-b}\phi|^2 - |\phi|^4 - \frac{1}{4\pi}b\,\mathrm{d}b + \frac{1}{2\pi}b\,\mathrm{d}A \leftrightarrow \mathrm{i}\bar{\Psi}\not{D}_{-A}\Psi + \frac{1}{4\pi}A\,\mathrm{d}A. \tag{2.8}$$

Charge conjugation operation over Ψ transforms the Dirac operator $\not{D}_{-A} \to \not{D}_A$. Consequently, we have

$$\mathrm{i}\bar{\chi}\not{D}_a\chi + \frac{1}{2\pi}a\,\mathrm{d}b - \frac{2}{4\pi}b\,\mathrm{d}b + \frac{1}{2\pi}b\,\mathrm{d}A \leftrightarrow \mathrm{i}\bar{\Psi}\not{D}_A\Psi + \frac{1}{4\pi}A\,\mathrm{d}A. \tag{2.9}$$

As we explained, the fermion path integral defined this way is T invariant up to an anomaly. We can attach a bulk term $-\frac{1}{8\pi}A\,\mathrm{d}A$ to both side to make it T invariant, leading to

$$\mathrm{i}\bar{\chi}\not{D}_a\chi + \frac{1}{2\pi}a\,\mathrm{d}b - \frac{2}{4\pi}b\,\mathrm{d}b + \frac{1}{2\pi}b\,\mathrm{d}A - \frac{1}{8\pi}A\,\mathrm{d}A \leftrightarrow \mathrm{i}\bar{\Psi}\not{D}_A\Psi + \frac{1}{8\pi}A\,\mathrm{d}A. \tag{2.10}$$

This theory in claimed to be the precise statement of the fermionic particle-vortex duality proposed in Ref. [9, 12, 13]. To demonstrate the the resemblance, let us try to integrate out b, whose equation of motion is $2\mathrm{d}b = \mathrm{d}(a + A)$. If we naively plug this relation into the Lagrangian, it results in

$$\mathrm{i}\bar{\chi}\not{D}_a\chi + \frac{1}{4\pi}a\,\mathrm{d}A + \frac{1}{8\pi}a\,\mathrm{d}a. \tag{2.11}$$

If we incorporate those $\frac{1}{8\pi}a\,\mathrm{d}a/\frac{1}{8\pi}A\,\mathrm{d}A$ into the definition of ermionic path integrals, the duality after eliminating b becomes

$$\mathrm{i}\bar{\chi}\not{D}_a\chi + \frac{1}{2\pi}a\,\mathrm{d}A \leftrightarrow \mathrm{i}\bar{\Psi}\not{D}_A\Psi, \tag{2.12}$$

which is the proposal in Ref. [9, 12, 13]. The reason that we had better not eliminate b so casually is $b = (a + A)/2$ doesn't satisfy proper flux quantization. To address this loophole, in Ref. [8] they suggest one should impose a different flux quantization condition for the field a

$$\int \frac{\mathrm{d}a}{(2\pi)} = 2\mathbb{Z} \tag{2.13}$$

and they point out this assumption is not innocuous as a is dynamical.

Let us discuss what a bit the operator mapping for this fermion particle-vortex duality.

Let us again borrow the argument in Ref. [8]. In the simpler picture (2.12), owing to (2.13), the smallest vortex f_M that we can have is a double vortex. There are two 0-modes transforming 1/2 representation of SU(2). The ground state $|0\rangle$, single 0-mode state and the double 0-mode state carry gauge charge $q_a = -1$, 0, and 1 respectively. Because a itself is not electrically charged under a, Gauss law demands we focus on the single 0-mode sector $q_a = 0$, and it thus corresponds spin 1/2 state. Thus, the physical electron Ψ, which is electrically charged 1 under A, corresponds to a neutral (under a) 0-mode χ and a double vortex f_M which is also electrically charged +1 under A.

We can also look at the correspondence in the refined picture (2.10), where we can consider vortex events of a and b. χ has electric charge $q_a = +1$. $\mathrm{d}a_M$ has electric charge $q_b = 1$ and $\mathrm{d}b_M$ has electric charge $q_a = 1$, $q_b = -2$ and $q_A = 1$. The simplest operator that is gauge invariant under a and b and charged +1 under A is

$$\chi^\dagger(\mathrm{d}a_M)^2(\mathrm{d}b_M). \tag{2.14}$$

This can be rephrased. As $\mathrm{d}A = 0$, let us consider a single vortex of b field, $\mathrm{d}b_M$. Gauss law obtained by $\frac{\delta}{\delta b^0}$, that

$$\frac{1}{2\pi}(\nabla\times\mathbf{a}) - 2\frac{1}{2\pi}(\nabla\times\mathbf{b}) = 0, \tag{2.15}$$

turns on a double vortex of a, while Gauss law from $\frac{\delta}{\delta a^0}$ turns on a fermion mode

$$\rho_\chi + \frac{1}{2\pi}\epsilon^{ij}\partial_i b_j = 0. \tag{2.16}$$

Interpreting this simple combination as physical electron gives the same result as (2.14). Thinking in terms of the 0-mode, this time fermion 0-mode states permitted transform under spin 0 representation since b has $q_a = 1$ as well. Then we go back to the discussion the hypothetical boson-fermion duality. The relative angular momentum between $\chi^\dagger$ and $\mathrm{d}b_M$ is 1/2 and thus the combination is a fermion. In the following discussion, we will often use the *naive* duality (2.12).

2.3 Self Dual Theories

Applying the fermion-fermion duality for 2 fermions reveals the self-dual structure of the following Lagrangian [4, 8, 15]

$$\mathscr{L}_{\mathrm{SD}}(A,B) = \mathrm{i}\bar{\psi}_1\gamma^\mu(\partial_\mu - \mathrm{i}(a+B)_\mu)\psi_1 + \mathrm{i}\bar{\psi}_2\gamma^\mu(\partial_\mu - \mathrm{i}(a-B)_\mu)\psi_2 + \frac{1}{2\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu A_\rho, \tag{2.17}$$

where ψ_is are 2-component spinors representing electrons, and A_μ and B_μ are both classical fields. This theory is self-dual in the sense that its partition function stays the same after interchanging A and B.

$$\mathcal{Z}[A;B] = \mathcal{Z}[B;A]. \tag{2.18}$$

That is, it is dual to the following theory

$$\bar{\mathscr{L}}_{\mathrm{SD}}(B,A) = \mathrm{i}\bar{\chi}_1\gamma^\mu(\partial_\mu - \mathrm{i}(b+A)_\mu)\chi_1 + \mathrm{i}\bar{\chi}_2\gamma^\mu(\partial_\mu - \mathrm{i}(b-A)_\mu)\chi_2 + \frac{1}{2\pi}\epsilon^{\mu\nu\rho}b_\mu\partial_\nu B_\rho, \tag{2.19}$$

where we use χs and b_μ to denote dual fermions and dynamical gauge field. We can actually provide a handy derivation here following the argument in Ref. [4]. First we apply (2.12) to (2.17), obtaining

$$\mathrm{i}\bar{\chi}_1\gamma^\mu(\partial_\mu - \mathrm{i}b_\mu)\chi_1 + \mathrm{i}\bar{\chi}_2\gamma^\mu(\partial_\mu - \mathrm{i}c_\mu)\chi_2 + \frac{1}{2\pi}a\,\mathrm{d}A + \frac{1}{2}\frac{1}{2\pi}b\,\mathrm{d}(a-B) + \frac{1}{2}\frac{1}{2\pi}c\,\mathrm{d}(a+B) \tag{2.20}$$

Terms linear in a can be collected and integrated out, imposing a constraint on other gauge fields (provided there is no holomony).

$$\frac{1}{4\pi}a\,\mathrm{d}(2A+c+b) \Rightarrow b = -c - 2A. \tag{2.21}$$

Consequently,

$$\mathrm{i}\bar{\chi}_2\gamma^\mu(\partial_\mu - \mathrm{i}c_\mu)\chi_2 + \mathrm{i}\bar{\chi}_1\gamma^\mu(\partial_\mu + \mathrm{i}(c+2A)_\mu)\chi_1 + \frac{1}{2\pi}(c+A)\,\mathrm{d}B. \tag{2.22}$$

Under a particle-hole transformation on χ_1, the gauge coupling obtains a minus sign. Then defining $c + A \to c$ completes the derivation of duality.

One may wonder if this self-dual theory survives after we *refine* the theory as (2.10). Actually there is also a refined counter part derived in the last section of Ref. [5].
This theory is intriguing in the sense that on each side of duality, only partial symmetry manifests. Without any external field, the gauged $N_\mathrm{f}^{(2)} = 2$ theory has explicit global SU(2) flavor symmetry, and a hidden U(1) symmetry coupled to the topological current $j = \epsilon\partial a$. Superficially, the symmetry encoded is SU(2)×U(1). To each continuous global symmetry, we turn on classical field coupling to the corresponding current. For example, in (2.17), A is coupled to U(1) current, while B couples to the Cartan generator of SU(2) current. On the dual side, the roles played by A and B get swapped. Schematically $\mathrm{U}(1)_A\times\mathrm{SU}(2)_B \leftrightarrow \mathrm{U}(1)_B\times\mathrm{SU}(2)_A$. Thus, if the duality holds, the full theory actually has a global symmetry group SU(2)×SU(2). This provides another explanation for the O(4) symmetry structure of 2-flavor QED_3 pointed out in Ref. [4, 11].
Actually, making use of the celebrated bosonic particle-vortex duality between abelian Higgs model and XY model, a bosonic self-dual theory containing $N_\mathrm{s} = 2$ can also be derived. A derivation at the level of partition function is given in [8]. Here we demonstrate a heuristic and handy version.
By bosonic particle-vortex duality we mean the equivalence of partition function between

$$\mathscr{L}_\mathrm{XY} = |(\partial_\mu - \mathrm{i}A_\mu)\phi|^2 \leftrightarrow \mathscr{L}_\mathrm{Higgs} = |(\partial_\mu - \mathrm{i}a_\mu)\varphi|^2 + \frac{1}{2\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu A_\rho. \tag{2.23}$$

Given this duality, one can further consider the following theory

$$\mathscr{L}_\mathrm{bSD}(A,B) = |(\partial_\mu - \mathrm{i}(a+B)_\mu)\phi_1|^2 + |(\partial_\mu - \mathrm{i}(a-B)_\mu)\phi_2|^2 + \frac{1}{2\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu A_\rho, \tag{2.24}$$

which under duality can be written

$$|(\partial_\mu - \mathrm{i}b_\mu)\varphi_2|^2 + |(\partial_\mu - \mathrm{i}c_\mu)\varphi_1|^2 + \frac{1}{2\pi}B\,\mathrm{d}(b-c) + \frac{1}{2\pi}a\,\mathrm{d}(b+c+A). \tag{2.25}$$

Integrating out a_μ imposes the constraint $b = -c - A$. Besides, we perform a charge conjugation transformation on φ_2 and redefine $c + A/2 \to c$. Consequently, the resulting Lagrangian is

$$|(\partial_\mu - \mathrm{i}(c - A/2)_\mu)\varphi_1|^2 + |(\partial_\mu - \mathrm{i}(c + A/2)_\mu)\varphi_2|^2 + \frac{1}{2\pi}c\,\mathrm{d}(-2B) = \mathscr{L}_\mathrm{bSD}(-2B, -A/2), \tag{2.26}$$

The resulting Lagrangian thus implies the duality relation

$$\mathcal{Z}[A;B] = \mathcal{Z}[-2B;-A/2]. \tag{2.27}$$

3 Other Issues

We have shown by assuming the boson-fermion duality, other dualities can be derived. Thus, one may ask what is behind this hypothetical theory. Actually, the whole story can be

incorporated in a large set of dualities called *level-rank* duality [1].
Another route is starting from Mirror symmetry. It has been shown the duality (1.1) can be derived by deforming $\mathcal{N} = 4$ mirror symmetry [6]. More phases are derived in [7].
Besides, regarding the self-dual stories stated in the last part, there is no free side, both being interacting without an apparent large or small parameter. Thus a reliable computation scheme is needed to calculate physical quantities, taking the dressed gauge boson propagator $\langle a_\mu(p) a_\nu(-p) \rangle$ for example.
A technique that is implemented in [3, 4], where they consider large charge limit. Together with dualities, they show in one description some fermions are completely decoupled in from any gauge fields in large charge limit and thus their current-current correlation functions can be computed without ambiguity. The current-current correlation function becomes gauge boson-boson correlation function on the other side of duality. This way they derived the fully dressed gauge boson propagator.
If we can develop techniques to perform calculation or discover so more applications to physical systems in terms of these dualities is what we are all looking forward to.

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