Physics 234: Quantum Mechanics

Homework 2.

1. $\hat{\boldsymbol{i}}$, $\hat{\boldsymbol{j}}$, and $\hat{\boldsymbol{k}}$ are unit vectors along the x, y, and z directions, respectively. Pauli matrices can be written in the vector form as $\vec{\sigma} = \sigma_1 \hat{\boldsymbol{i}} + \sigma_2 \hat{\boldsymbol{j}} + \sigma_3 \hat{\boldsymbol{k}}$. The unit vector in a general direction is $\vec{n} = \sin\theta\cos\phi\hat{\boldsymbol{i}} + \sin\theta\sin\phi\hat{\boldsymbol{j}} + \cos\theta\hat{\boldsymbol{k}}$. Find the eigenvalues and eigenstates of $\vec{\sigma}\cdot\vec{n} = \sin\theta\cos\phi\sigma_1 + \sin\theta\sin\phi\sigma_2 + \cos\theta\sigma_3$.

2. Consider the following commutator, $[\sigma_1, \vec{\sigma}\cdot\vec{a}] = (\sigma_1(\vec{\sigma}\cdot\vec{a}) - (\vec{\sigma}\cdot\vec{a})\sigma_1)$, where $\vec{a}$ is an arbitrary vector. Prove $[\sigma_1, \vec{\sigma}\cdot\vec{a}] = (2i\vec{a}\times\vec{\sigma})\cdot\hat{\boldsymbol{i}}$. Note that "$\times$" denotes the *cross product* of two vectors.

3. Define the exponential of an operator O to be

$$e^O = \sum_{n=0}^{\infty} \frac{1}{n!} O^n = 1 + O + \frac{O^2}{2} + \ldots$$

Prove

$$e^{i\sigma_i\theta} = \cos\theta + i\sigma_i\sin\theta$$

where σ_i is any one of the Pauli matrices.