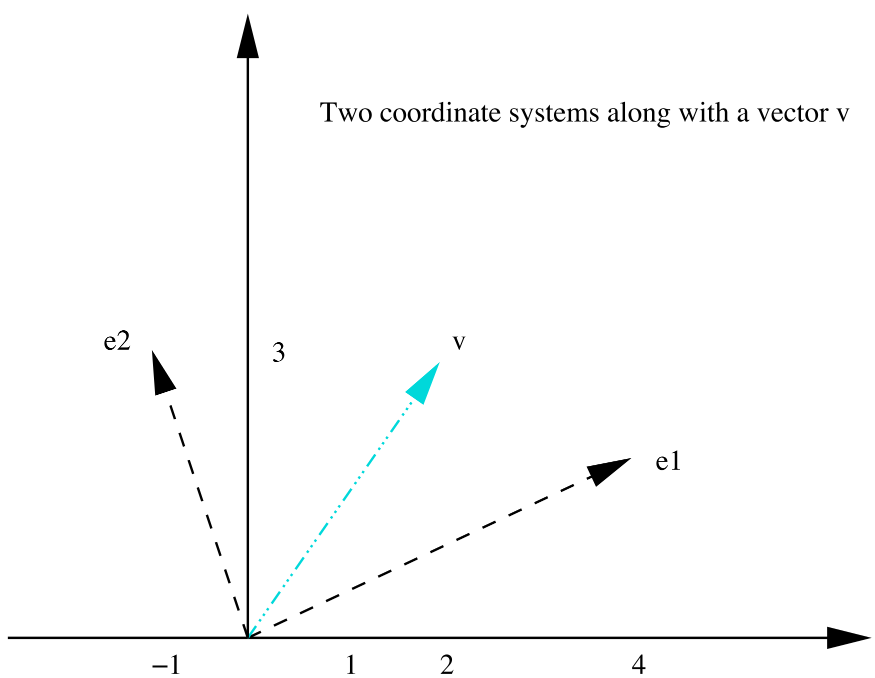
Notes on Matrices and Linear Transformations

Physics 221

Let's recall that vectors are objects independent of which coordinates we choose to describe them. For example, the vector v depicted below exists regardless of the coordinates that you or I might choose to describe it.



So let's imagine two coordinate systems. The first is the familiar $\hat{\imath}, \hat{\jmath}$ system where

$$\hat{\imath} = (1,0), \quad \hat{\jmath} = (0,1). \tag{1}$$

With respect to this basis, v is described by coordinates $(2,3)$ or

$$v = 2\hat{\imath} + 3\hat{\jmath}. \tag{2}$$

On the other hand, an equally good basis is given by the two linearly independent vectors e_1, e_2. What are the coordinates in this basis? To answer this question, we need to determine the linear transformation S that relates the two bases. First we observe that

$$e_1 = 4\hat{\imath} + 2\hat{\jmath}, \quad e_2 = -\hat{\imath} + 3\hat{\jmath}. \tag{3}$$

Note that the dot product $e_1 \cdot e_2 = -4 + 6 = 2$ so this basis does not consist of orthogonal vectors unlike our starting basis. You can see this in the picture.

We can also invert this relation to solve for $(\hat{\imath}, \hat{\jmath})$ in terms of (e_1, e_2):

$$\hat{\imath} = \frac{3e_1 - 2e_2}{14}, \quad \hat{\jmath} = \frac{4e_2 + e_1}{14}. \tag{4}$$

What this teaches us is that a general vector expressed in the $(\hat{\imath}, \hat{\jmath})$ basis has coordinates

$$v = \alpha_1 \hat{\imath} + \alpha_2 \hat{\jmath} \tag{5}$$

which we can express in the (e_1, e_2) basis using (4) as follows

$$\begin{aligned} v &= \alpha_1 \hat{\imath} + \alpha_2 \hat{\jmath} = \alpha_1 \frac{3e_1 - 2e_2}{14} + \alpha_2 \frac{4e_2 + e_1}{14} \\ &= \left(\alpha_1 \frac{3}{14} + \alpha_2 \frac{1}{14} \right) e_1 + \left(-\alpha_1 \frac{2}{14} + \alpha_2 \frac{4}{14} \right) e_2 \\ &= \beta_1 e_1 + \beta_2 e_2. \end{aligned} \tag{6}$$

This relates the α coordinates to β coordinates with respect to the new basis.

Let's write this as a linear transformation S acting on α to give β

$$\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}. \tag{7}$$

This change of basis matrix determines everything. First it must have non-zero determinant since it maps a basis to a basis and hence must be invertible. It's easy to check that $\det(S) = 1/14$. This means that S is an element of the group $GL(2, \mathbb{R})$ of invertible 2×2 matrices.

The inverse is also easy to determine either by using the general form for the inverse of a 2×2 matrix or by inspection:

$$S^{-1} = \begin{pmatrix} 4 & -1 \\ 2 & 3 \end{pmatrix}. \tag{8}$$

So v is described with respect to the α coordinates by $(2, 3)$ but using (7) we see that v is also described in the β coordinates by $(9/14, 8/14)$.

Once you have S, you can determine the matrix representing a linear transformation in this new basis. For example, suppose the linear transformation T is represented by a 2×2 matrix M in the $(\hat{\imath}, \hat{\jmath})$ basis so that

$$T \cdot v \quad \rightarrow \quad M \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}. \tag{9}$$

Well, we can determine how T is represented in the (e_1, e_2) basis by inserting the identity matrix in a clever way:

$$\begin{aligned} M \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} &= M \left(S^{-1} S \right) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \left(M S^{-1} \right) S \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \\ &= \left(M S^{-1} \right) \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}. \end{aligned} \tag{10}$$

Now this still gives the output result with respect to the α coordinates which we need to convert to β coordinates using (7). This amounts to another application of S so with respect to the (e_1, e_2) basis, we see that T is represented by

$$T \quad \rightarrow \quad S M S^{-1}. \tag{11}$$

This is conjugation of T by S. Conjugation amounts to a change of basis which is why we are interested in the issue of whether matrices can be diagonalized by conjugation.