Problem Set 2

Physics 221

Due October 15

Some abbreviations: B - Boas.

1. Let's start with by some practice with vector spaces. Show that the definition of a quotient space V/W of two vector spaces V and W given in lecture, namely the space of equivalence classes $[v]$ where $[v_1] \sim [v_2]$ if $v_1 - v_2 \in W$, satisfies the axioms of a vector space. Compute the dimension of V/W.

Take $V = \mathbb{R}^2$ and W to be the 1-dimensional vector space containing the vector $(1, \sqrt{2})$. Can you graphically depict V/W?

2. Take $\mathbb{R}^3$ with a basis,

$$\vec{e}_1 = (1,0,0), \qquad \vec{e}_2 = (1,1,0), \qquad \vec{e}_3 = (1,1,1).$$

Using this basis, we can expand vectors $\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + v_3\vec{e}_3$. Consider the linear transformation T that acts on $\vec{v}$ as follows:

$$T : \vec{v} \to M\vec{v} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 3 \\ a & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Here a is a real number.

(i) What is the rank of T? Does this depend on the choice of a? What is the kernel of T? What is the $\mathrm{cok}(T)$? What is $\det(T)$? What is $\mathrm{Tr}(T)$? When is this map an isomorphism? Prove your claim.

(ii) We could also express $\vec{v}$ in terms of the usual $\hat{\imath}, \hat{\jmath}, \hat{k}$ basis where $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$. What is the matrix representation of T in this basis? Let's call this matrix M'. Compute $\det(M')$ and $\mathrm{Tr}(M')$.

(iii) These two basis choices are also related by a linear transformation. Construct the matrix, call it S, that takes you from the first basis to the second. Is S invertible? If so, construct the inverse and compute the determinant.

(iv) Find a relation between M, M' and S.

(v) We discussed the definition of and some examples of groups in lecture. Now the set of invertible linear transformations of a vector space V parametrize all possible choices of bases. You can get from one basis to another by some invertible linear transformation. To get some practice with groups, show that these invertible transformations form a group. This group is called $GL(V)$ which stands for "general linear transformations of V". Show that if $\dim(V) > 1$, this group is non-abelian.

3. There is a beautiful and fundamental set of matrices in physics called the Pauli matrices. The reason for their importance is that they are used to describe how spatial rotations in three dimensions act on fermions in nature; for example, on electrons.

(i) These matrices are given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Show that these are Hermitian matrices. Show that the commutator $[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$.

(ii) The anti-commutator of two operators or matrices is defined as follows:

$$\{A, B\} = AB + BA.$$

Show that $\{\sigma^i, \sigma^j\} = 2\delta^{ij}\mathbf{1}$.

4. We defined the group $U(1)$ as unitary 1 by 1 matrices that preserve the norm $||z||^2 = z^*z$. Let's extend this in the following way. Consider $\mathbb{C}^2$ with vectors $\vec{z} = (z_1, z_2)$ and a norm $||\vec{z}||^2 = z_1^*z_1 + z_2^*z_2$. The set of matrices that preserve this norm form a group called $U(2)$. Show that $U(2)$ consists of unitary 2 by 2 matrices.

5. B. p.122 #9 & #15 & #17 & #18 & #22 & #30 & #32

6. B. p.136 #4

7. B. p.141 #3