Problem Set 5

Physics 221

Due November 5

Some abbreviations: B - Boas.

1. B. p.363 #10

2.(i) Consider periodic functions on the interval $[0, 2\pi]$. Find the Fourier expansion for $f(x) = \cos(2x)\sin(5x)$ and for $f(x) = \sin(3x - \phi)$ for some constant ϕ.

(ii) Now we claimed in lecture that there are really only two kinds of Hilbert space up to isomorphism; either $\mathbb{C}^N$ or the space ℓ_2. If this is true then the space of normalizable periodic functions on the interval $[0, 2\pi]$ should be isomorphic to either $\mathbb{C}^N$ or the space ℓ_2. Which of the two is it and why? To show this equivalence, recall that ℓ_2 consists of all vectors $(x_1, x_2, \ldots)$ such that $|x_1|^2 + |x_2|^2 + \ldots < \infty$.

Consider a periodic function expanded in a nice orthonormal basis

$$|f> = \sum_n c_n |n>$$

following the notation used in lecture. Consider the vector constructed from the Fourier coefficients $(\ldots, c_{-1}, c_0, c_1, c_2, \ldots)$. Show that the condition that $|f>$ has finite norm implies that this vector is in ℓ_2. Use bra-ket notation. By doing so, you have essentially constructed the desired isomorphism. Every vector in ℓ_2 corresponds to a normalizable periodic function and vice-versa.

3. (i) Let's start with an exercise involving Gram-Schmidt. Take the following three vectors in $\mathbb{R}^3$,

$$e_1 = (1, 1, 1), \quad e_2 = (1, 1, 0), \quad e_3 = (1, 0, 1).$$

Construct an orthonormal basis using the Gram-Schmidt procedure starting with e_1.

(ii) Now let's return to normalizable functions on the interval $[-1, 1]$ expanded in the basis $\{1, x, x^2, x^3, \ldots\}$. In lecture, we took a weight function $w(x) = 1$ and we constructed the first three elements of an orthogonal basis which were related to Legendre polynomials:

$$\phi_0 = \frac{1}{\sqrt{2}}, \quad \phi_1 = \sqrt{\frac{3}{2}}x, \quad \phi_2 = \sqrt{\frac{5}{2}}\frac{3x^2 - 1}{2}.$$

Construct ϕ_3 and ϕ_4.

(iii) Lastly, let us change the weight function to $w(x) = e^{-x^2}$ and let us consider functions from $(-\infty, \infty)$. For these functions to be normalizable, they must grow sufficiently slowly as $|x| \to \infty$ to ensure that their norm is finite. Is x^2 a normalizable function with respect to this inner product? Compute the norm. Again starting with the basis $\{1, x, x^2, x^3, \ldots\}$, construct the first 3 elements of an orthonormal basis i.e., $\{\phi_0, \phi_1, \phi_2\}$. These orthogonal polynomials are called Hermite polynomials and they play an important role in the study of simple harmonic oscillators like springs.