Quantum Games and Game Strategy

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In this paper, we will underscore the basics of classical computing / game strategy and the basics of quantum computing needed to explore the effects that quantum information exchange has on classical games. We will then discuss two examples of quantized games: the PQ penny flip game, which is considered to be the first quantized game, and a quantized Card Game problem.

Introduction

In 1999, D. A. Meyer merged the subject of game theory with the new and exciting world of quantum computing, introducing the world to the first quantum, or quantized, game. Since 1999, many mathematicians and physicists alike have added to this new class of games, developing more quantum games and quantum game theory. Quantum game theory differs from classical game theory in three main ways: the initial states can be superimposed, the initial states can be quantum entangled, and a superposition of strategies can be acted on the initial states. Most of the time, this manifests in quantum games as one or more of the players employing quantum strategies by acting quantum gates, using measurement tools only available in quantum mechanics, or manipulating the system through uniquely quantum means. It is important to note that while both classical and quantum games may have real world applications in the form of information theory and other computing based fields, what often makes them interesting is not just their applications, but rather the intellectual puzzle one explores in working them out.

Background Information

1. Classical Computing

In order to understand the revolution that is quantum computing, we must first understand the basics of classical computing. On any given standard computer, the most basic unit of information is the bit. The bit in its most simple form denotes a 0 or 1, but depending on the situation can also be representative of true or false, on or off, yes or no, or another pair of binary information.

1.1 Classical Gates

A classical gate is an operator which acts on an ordered input sequence of k bits $(b_1,b_2,...,b_k)$, with $k \geq 1$, and the results are ordered in an output sequence of l bits $(\beta_1,\beta_2,...,\beta_l)$ such that:

$$G(b_1,b_2,...,b_k) = \beta_1,\beta_2,...,\beta_l \tag{1}$$

where $b_i,\beta_i = 0,1$. The most common classical gates are as follows: the NOT, OR, AND, NAND, NOR, and XOR gates. See (10) for diagrams and descriptions of these gates.

2. Quantum Computing

In quantum computing, a qubit is an elementary unit, often representing a microscopic system such as nuclear spin or a polarized photon. For qubits, the states 0 or 1 are prescribed by one of two normalized, mutually orthogonal, states which we label as $\{|0\rangle,|1\rangle\}$. We define these states such that:

$$|1\rangle = \begin{pmatrix}0\\1\end{pmatrix} \quad |0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \tag{2}$$

These two states form our basis.

2.1 Pure States and Superpositions

We can represent the state of our qubit in one of two ways; as in a classical state or as in a superposition such that:

$$\begin{aligned} |\psi_{classical}\rangle &= \gamma|n\rangle, \quad n=0,1 \quad |\gamma|^2=1 \\ |\psi_{superposition}\rangle &= \alpha|0\rangle+\beta|1\rangle \quad |\alpha|^2+|\beta|^2=1 \end{aligned} \tag{3}$$

It is important to note that when performing a calculation on a superposition, the calculation is performed on all pure states composing the superposition. Namely, in an n qubit superposition, the calculation is performed over the 2^n pure states in that superposition. Finally, when measuring or observing a qubit in a superposition, the superposition collapses to that state, allowing one to only measure the pure state.

2.2 Density Matrices

Another important piece of formalism necessary for our discussion of quantum games is the density matrix and its spectral decomposition. Instead of using $|0\rangle$ and $|1\rangle$ for pure states, or weighted sums of $|0\rangle$ and $|1\rangle$ for mixed states, to represent our system, we can also use positive, semidefinite, Hermitian matrices with trace 1 to describe the state of our system. We call these density matrices. Any general density matrix, ρ, can be written as the weighted sum of pure states, $|\phi\rangle\langle\phi|$ such that:

$$\rho = \sum_n \lambda_n|\phi_n\rangle\langle\phi_n| \quad \text{where} \quad \lambda_n \geq 0 \quad \text{and} \quad \sum_n \lambda_n = 1 \tag{4}$$

Here, λ_n is the probability of getting the state $|\phi_n\rangle$. The density matrix is not a completely different conception of our system, but rather a re-framing of what we have already discussed! For example, where before pure and mixed states were described by equation (3), they can also be described by the following density matrices:

$$\begin{aligned} \rho_{\ pure} &= \lambda_i|\phi_i\rangle\langle\phi_i| \quad \text{where} \quad \lambda_i = 1 \\ \rho_{\ mixed} &= \sum_n \lambda_n|\phi_n\rangle\langle\phi_n| \quad \text{where} \quad \sum_n \lambda_n = 1 \end{aligned} \tag{5}$$

Finally, it is easy to see that if ρ is comprised of pure states denoted by $|\phi_i\rangle$, then the λ_i's in ρ denote the probability of that pure state! This notion is also motivated by spectral decomposition, which is important to the construction of the density matrix. For more information on spectral decomposition, see reference, (21).

2.3 Unitary Matrices

In Quantum game theory, it is important to note that if any manipulation of the qubits is needed, it must be performed by unitary operations. A unitary matrix, $\mathbf{U}$, is a matrix whose adjoint, $\mathbf{U}^\dagger$, is equivalent to its inverse, $\mathbf{U}^{-1}$ such that:

$$\mathbf{U}^\dagger\mathbf{U} = \mathbf{U}\mathbf{U}^\dagger = \mathbf{I} \tag{6}$$

These unitary transformations function as gates do in classical computing. For example, the identity matrix preserves the current state.

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7}$$

The X Pauli matrix performs a NOT operation, switching a pure state 1 to a 0 and vice versa.

$$\mathbf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{8}$$

The Z Pauli matrix introduces a phase by switching the sign on the state 1.

$$\mathbf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{9}$$

And perhaps the most important unitary matrix for our discussion of quantum gaming is the Hadamard matrix, which transforms a pure state $|0\rangle$ into a superposition with equal probability of being in state $|0\rangle$ and state $|1\rangle$.

$$\mathbf{H} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{10}$$

It is important to note that acting the Hadamard transform on a pure state, $|0\rangle$ or $|1\rangle$, puts the system into a superposition state such that:

$$\mathbf{H}|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{11}$$

$$\mathbf{H}|1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{12}$$

Or in the Hadamard basis,

$$\mathbf{H}|0\rangle = |+\rangle \quad \text{where} \quad |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{13}$$

$$\mathbf{H}|1\rangle = |-\rangle \quad \text{where} \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{14}$$

Furthermore, acting the Hadamard transform on a superposition state which is already in the Hadamard basis returns a pure state such that:

$$\mathbf{H}|+\rangle = |0\rangle \tag{15}$$

$$\mathbf{H}|-\rangle = |1\rangle \tag{16}$$

This is an important aspect of the Hadamard gate which will become useful later.

3. Classical Game Theory

3.1 Classical Strategies

In classical game theory, players can employ one of two types of strategies: pure or mixed strategies. For example, let's say there exists a game in which the possible set of actions is prescribed by a_i. Then the player can choose to use either strategy S_{pure} or S_{mixed} in which the probability of choosing action a_i is P_i such that:

$$\begin{aligned} S_{pure} &= P_j a_j \quad \text{where} \quad P_j = 1 \\ S_{mixed} &= \sum_i P_i a_i \quad \text{where} \quad \sum_i P_i = 1 \end{aligned} \tag{17}$$

Or in other words, a pure strategy is deterministic and mixed strategy is probabilistic.

3.2 Nash Equilibrium

Another important piece of information we need for our discussion of quantum games is the classical concept of Nash Equilibrium. In simple terms, the players of a game are considered to be in Nash Equilibrium if player A does not gain anything from deviating from their initial strategy, assuming player B also leaves their strategy unchanged.

In order to describe the concept of a Nash Equilibrium more formally, we must first understand the idea of an action profile. In game theory, the action profile is a list of actions for each player in the game with every list item corresponding to an individual player. In other words, the action profile is a comprehensive description of the action done by each player in the game.

Formally, as described by Osborne (19), let us define the action profile a such that a is the set of the actions a_i performed by i players. Let $a_i^{'}$ denote the action of every player i (either $= a_i$ *or* $\neq a_i$). Then $(a_i^{'}, a_{-i})$ denotes the action profile in which every player j, except player i, chooses their action to be $a_j \in a$ whereas player i chooses their action to be $a_i^{'} \notin a$.

Then the action profile a^* is a Nash Equilibrium if, for every player i with action a_i,a^* is at least as preferable as the action profile (a_i, a_{-i}^*). In other words, player $i's$ payoff function, u_i is such that:

$$u_i(a^*) \ \geq \ u_i(a_i, a_{-i}^*) \ \forall \ a_i \tag{18}$$

Where any player's payoff function describes the award given to that player at the outcome of the game. For example, if I were to bet 1 dollar that a flipped penny would land heads up, my payoff function would be a function of the two outcomes, heads up and heads down, equalling +1 if the penny were to land heads up and -1 if the penny were to land heads down. The payoff function is important because it encodes important information about the game at hand–what each player wins, or gains, given a specific action profile.

It is important to note that a player's payoff function, u_i is different from that player's expected payoff, $\overline{u}_i$. The expected payoff is the sum of all payoff possibilities weighted by the probability that payoff will occur. The expected payoff is also important as it can be a measure of whether or not a game is *fair.* If player A's expected payoff for a game is +100 and player B's expected payoff for that same game is -100, perhaps player B should play a different game...

3.3 Zero Sum Games and Non-Local Games

The last piece of information necessary for our discussion of classical game theory is the concept of a zero-sum game. A zero-sum game is a two-player game in which one player can only make themselves better off by making the other player worse off. Or in other words, a game in which the payoff function of player A is the negative of the payoff function of player B such that:

$$u_A = -u_B \tag{19}$$

The unique Nash Equilibrium of most classical zero-sum games is effectively a draw. For example, in tic-tac-toe, if player A is playing her best and player B is equally playing her best, neither player can do better for herself than to draw, if both players are trying to win and trying not to lose. Zero-sum games have a unique Nash Equilibrium because if there were a way for player A or player B to improve her strategy, the game would not be zero sum. In other words, player B is forced to play for a draw if player A correctly plays to win in tic-tac-toe, or else player B loses.

Finally, it is important to note that the majority of–if not all–classical games that can be quantized belong to a class of classical games called non-local games. This type of game is a game in which the players are separated by time or space such that every player acts their move with no knowledge of the moves being performed by any other player. Essentially, in non-local games, the players are playing a cooperative game of incomplete information.

4. Quantum Games

As mentioned in the introduction, quantum games are exciting thought experiments in which one allows one or more players in a classical game to employ quantum strategies. The act of quantizing classical non-local games can produce quantum games which have drastically changed the odds or outcome of their classical counterpart, either giving supreme advantage to a player in a classically fair game, or giving a player the proper tools to even the odds in a classically unfair game.

Since Meyer introduced the first quantized game in 1999, numerous new quantum games with varying levels of complexities and outcomes have been written about by anyone from physicists to statisticians to game theorists. The following paragraphs will discuss how the outcomes of two classical games–the PQ penny flip game and a classical card game–change under the introduction of quantum strategy.

5. PQ Penny Flip Game

The classical version of Meyer's PQ penny flip game is comprised of the following four steps:

1. The referee places a penny heads up in covered a box such that neither P nor Q can see the state of the penny until the referee reveals it at the end of the game, but both P and Q know that the penny's initial state was heads up.
2. On Q's turn, she can either flip the penny (F) or do nothing (N).
3. On P's turn, she can also (F) or (N).
4. Q takes the final move either flipping the penny or not

At the end of Q's turn, the box is opened and the state of the penny is revealed. If the coin is revealed to be heads-down, P wins with payoff, $u_P = +1$, and Q loses with payoff, $u_Q = -1$. If the penny is revealed to be heads-up, Q wins with payoff, $u_Q = +1$ and P loses with payoff $u_P = -1$. We can summarize the classical penny flip game in the following table:

Move and Payoff Summary			
Q's 1st Move	P's Move	Q's 2nd Move	(u_Q,u_P)
N	N	N	(+1,-1)
N	F	N	(-1,+1)
N	N	F	(-1,+1)
N	F	F	(+1,-1)
F	N	N	(-1,+1)
F	F	N	(+1,-1)
F	N	F	(+1,-1)
F	F	F	(-1,+1)

So, out of the eight different combinations of moves, P and Q both have four ways of winning +1 point and four ways of losing -1 point. And when P wins +1, Q loses -1, and vice versa, such that their payoff functions can be summarized as:

$$u_P = -u_Q$$

This is the very definition of a zero-sum game! Classically, this is a fair game that sees both P and Q winning with probability, $P = 0.5$.

Finally, if we allow players P and Q to employ mixed strategies, it can be easily shown the mixed-strategy Nash Equilibrium of this game has player P chose (N) or (F) with equal probability, 0.5, and player Q choose between her four options, (NN), (FF), (NF), (FN), with equal probability 0.25. such that their strategies can be written as:

$$\begin{aligned} S_P &= 0.5\ (N) + 0.5\ (F) \\ S_Q &= 0.25\ (NN) + 0.25\ (FF) + 0.25\ (NF) + 0.25\ (FN) \end{aligned} \tag{20}$$

In this situation, player P and Q both have expected payoffs:

$$\begin{aligned} \overline{u}_P &= 0 \\ \overline{u}_Q &= 0 \end{aligned} \tag{21}$$

5.1 Quantized Penny Flip

In order to quantize this game, we will allow player Q free reign of quantum strategies, but restrict player P to classical strategies. For simplicity we will restrict P to pure classical strategies, but will later expand our results to classical mixed strategies as well. If we let $|0\rangle$ represent heads and $|1\rangle$ represent tails, then the state of the game at any point can be described by a quibit, $|\psi\rangle$, such that:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{22}$$

Where the initial state of the game is:

$$|\psi_{initial}\rangle = |0\rangle \tag{23}$$

The two classical moves, (F) and (N), can be described by the unitary transforms **X** and **I** respectively. While player P's is restricted to classical moves modeled by the above unitary matrices, player Q, on the other hand, is able to enact any unitary transform on her turns. Perhaps the most interesting of which is the Hadamard matrix, **H**.

Let's consider the situation where player Q plays the Hadamard matrix on both of her turns. After Q's first turn, the state of the game can be described as:

$$\mathbf{H}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$$

By playing **H**, Q has put the system into a superposition. Physically, she has placed the penny on its side. As a result of this quantum move, if P leaves the penny alone, **I**, or if P flips the penny, **X**, the state of the system remains unchanged. Explicitly:

$$\mathbf{I}|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = |+\rangle \tag{24}$$

$$\mathbf{X}|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = |+\rangle \tag{25}$$

On the final turn of the game, Q acts the Hadamard transform again, effectively undoing her original move, as shown in section 2.3:

$$\mathbf{H}|+\rangle = |0\rangle$$

Now, to summarize the two possible situations where Q plays the Hadamard matrix on both of her turns and P either does nothing, **I**, or flips the penny, **X**:

$$\mathbf{H}\ \mathbf{I}\ \mathbf{H}\ |0\rangle = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle \tag{26}$$

$$\mathbf{H}\ \mathbf{X}\ \mathbf{H}\ |0\rangle = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle \tag{27}$$

Therefore, if player P is restrained to classical strategies, and player Q is allowed to use quantum strategies, playing **H** on both of her turns, no matter what move player P plays, at the end of the game, the referee will always reveal the penny to be heads up, $|0\rangle$. Therefore, Q will always win with expected payoff $\overline{u}_q=1$ and P will always lose with expected payoff $\overline{u}_p=-1$. The quantization of Q's strategy puts the game in a state of superposition that cannot be altered by P, guaranteeing Q's victory.

5.2 Mathematical Motivation behind Selection of the Hadamard Matrix

Now, we will relax our restrictions a bit by allowing P to employ mixed classical game strategies instead of only pure strategies. We will also show that the selection of **H** is motivated by finding the Nash Equilibrium given Q acts a general unitary matrix. Let player P's mixed-strategy be to flip, **X**, with probability p, and not to flip, **I**, with probability 1-p. Again Q can employ quantum moves. Following Meyer (1) and others (3), let Q's transform be given by a unitary matrix of the form:

$$\mathbf{U}_{Q1}=\begin{bmatrix} a & b^* \\ b & -a^* \end{bmatrix} \quad \text{where} \quad aa^*+bb^*=1 \tag{28}$$

Since the game now involves mixed states, we must use a density matrix to describe the system. The initial state is:

$$\rho_0=|0\rangle\langle 0|=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{29}$$

After Q's first move, the state of the system is:

$$\rho_1=\mathbf{U}_{Q1}\rho_0\mathbf{U}_{Q1}^\dagger=\begin{bmatrix} aa & ab^* \\ ba^* & bb^* \end{bmatrix} \tag{30}$$

After P's move, the state of the system is:

$$\rho_2=p\mathbf{X}\rho_1\mathbf{X}^\dagger+(1-p)\mathbf{I}\rho_1\mathbf{I}^\dagger=p\begin{bmatrix} bb^* & ba^* \\ ab^* & aa^* \end{bmatrix}+(1-p)\begin{bmatrix} aa^* & ab^* \\ ba^* & bb^* \end{bmatrix} \tag{31}$$

As discussed in section 2.2, the diagonal elements of the density matrix correspond to the probability of pure states. Therefore, after P's turn, the probability that the penny is heads up, $|0\rangle$, or heads down, $|1\rangle$, can be expressed as follows:

$$\begin{aligned} Prob\ (|0\rangle)&=pbb^*+(1-p)aa^* \\ Prob\ (|1\rangle)&=paa^*+(1-p)bb^* \end{aligned} \tag{32}$$

So what does this tell us about P and Q's strategies? Well, P wants the penny to be revealed to be tails up. So, it is in P's best interest to maximize $Prob\ (|1\rangle)$ and minimize $Prob\ (|1\rangle)$. Therefore, given Q's strategy, if $aa^* > bb^*$ P's best response is to choose p=1, if $aa^* > bb^*$, P's best response is to choose p = 0, and finally, if $aa^* = bb^*$ P's best response is to choose any p $\in$ [0,1]. It seems somewhat trivial to call choosing any p $\in$ [0,1] a *best* response in the case that $aa^* = bb^*$, as the probability of the penny being heads up is equivalent to the probability of the penny being heads down and it is independent of P's choice of p.

Player Q, on the other hand, wants the penny to be revealed to be heads up. It is therefore in her best interest to maximize $Prob\ (|0\rangle)$ and minimize $Prob\ (|1\rangle)$, the inverse of what player P wanted to do. So, if p $>$ 0.5, it is in Q's best interest to choose $bb^* = 1$, if p $<$ 0.5 it is in Q's best interest to choose $bb^*=0$, and if p = 0.5 it is in Q's best interest to choose any a,b. Again, the last situation seems somewhat trivial as penny has an equal probability of being heads up and heads down which is independent from Q's choice of a and b.

But these seemingly trivial situations are, in fact, not trivial at all when it comes to deducing the Nash Equilibrium of this game. Therefore, it is clear that the Nash Equilibrium is p = 0.5 and $aa^* = bb^* = 0.5$ with both players getting payoffs $\overline{u}_q = \overline{u}_p = 0$. This result is exactly the same as the classical mixed-strategy equilibrium for both players, suggesting that a player with optimal quantum strategy has an expected payoff, u, at least as great as her expected payoff with optimal mixed-strategy.

Now, so far we have only worked out the general form for the first two moves of the game given P employs a mixed classical strategy and Q acts a general unitary matrix on her turns. We *could* go on to calculate the general state of the system at the conclusion of the game, ρ_3, but the algebra at that point becomes more messy than enlightening.

Now supposed that on Q's second move, she chooses a strategy with $a = b = 1/\sqrt{2}$ such that $aa^* = bb^* = 0.5$. Or in other words, she chooses the strategy where she enacts the Hadamard matrix on both of her turns such that:

$$\mathbf{U}_{Q1} = \mathbf{U}_{Q2} = \mathbf{H} \tag{33}$$

Then,

$$\rho_1 = \mathbf{U}_{Q1}\rho_0\mathbf{U}_{Q1}^{\dagger} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{34}$$

$$\rho_2 = p\mathbf{X}\rho_1\mathbf{X}^{\dagger} + (1-p)\mathbf{I}\rho_1\mathbf{I}^{\dagger} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{35}$$

$$\rho_3 = \mathbf{U}_{Q2}\rho_2\mathbf{U}_{Q2}^{\dagger} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{36}$$

And again, the diagonal terms of the density matrix correspond to the probability of pure states. Therefore, if player Q plays the Hadamard matrix on both of her turns, resulting in ρ_3, when the referee reveals the penny, it will always be heads up with probability 1 and will never be heads down with probability 0. Therefore, if player Q plays the Hadamard matrix on both of her turns, then she will always win with payoff $u_q = 1$, whether P employs pure or mixed quantum strategies.

So, it certainly seems that quantizing the PQ Penny flip game by allowing player Q to utilize quantum strategies while restricting P to classical strategies, creates an unfair situation in which player Q can always win by choosing to act the Hadamard matrix on both of her turns.

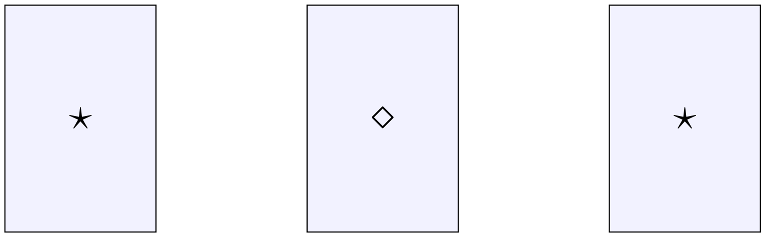
6. Alice and Bob Play Cards

However, Quantum game theory doesn't always guarantee that the player employing quantum strategies will always win. In fact, quantizing some classically unfair games can actually make them fair. In the following section we will discuss an example game in which allowing Bob to use a quantum query machine gives him fair odds against Alice where classically, he would have a 2/3 chance of losing.

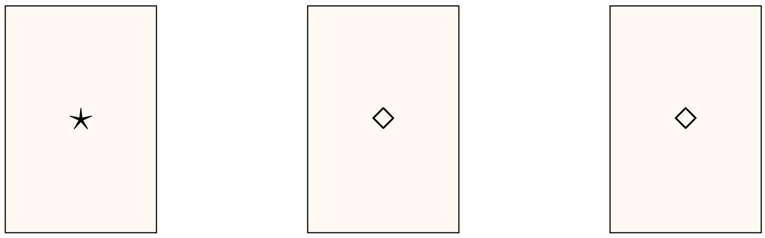
6.1 The Classical Card Game

Let's go over the classical version of this Alice–Bob card game. Let's say Alice and Bob play a card game. And let's say Alice has three cards which are identical except for the following markings: the first has a star on both sides, the second has a diamond on both sides, and the last has a star on the front but a diamond on the back.

Front of Cards



Back of Cards



Then let's say that Alice puts these three cards into a black box, shaking it up to randomize their position and topside. Then we let Bob draw a card from the box, without flipping it, such that both players can see the upper side of the card. If Bob draws a card with identical markings on both sides, Alice wins with payoff, $u_A = +1$ and Bob loses with payoff, $u_B = -1$. However, if Bob draws the card with different markings on front and back, Bob wins with payoff, $u_B = +1$ and Alice looses with payoff, $u_A = -1$. It is easy to see that in this situation, their expected payoffs are:

$$\begin{aligned} \overline{u}_A &= \frac{2}{3}(+1) + \frac{1}{3}(-1) = \frac{1}{3} \\ \overline{u}_B &= \frac{1}{3}(+1) + \frac{2}{3}(-1) = -\frac{1}{3} \end{aligned} \tag{37}$$

And that their probabilities of winning are:

$$P_A = \frac{2}{3} \quad \text{and} \quad P_B = \frac{1}{3} \tag{38}$$

It is clear that Bob is at a disadvantage in this situation and the game is hence unfair to him. So let's say that in order to attract Bob into playing with her, Alice gives Bob the chance to operate on the cards–ie Bob has one query on the black box–and then allows him to withdraw from the game if he'd like. Classically, Bob can only attain one card's information after his query. And therefore, the game is still unfair.

6.2 Quantized Card Game

Now suppose in the quantum representation, we describe state of the card with a diamond by $|0\rangle$ and the state of the card with a star by $|1\rangle$. Then if Alice prepares the system randomly as she did before, the upper sides of the cards in the box can be described by the following qubit, $|r\rangle$, such that:

$$|r\rangle = |r_0 \; r_1 \; r_2\rangle \quad \text{where} \quad r_k \in \{1,0\} \tag{39}$$

Next, suppose Bob has a quantum query machine that depends on state $|r\rangle$ in the black box. Suppose he constructs his query machine as follows. He starts with the unitary matrix, $\mathbf{U}_k$ such that:

$$\mathbf{U}_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi r_k} \end{bmatrix} \tag{40}$$

If $r_k = 0$, then $\mathbf{U}_k = \mathbf{I}$, but if $r_k = 1$, then $\mathbf{U}_k = \mathbf{Z}$ He then sandwiches this unitary matrix between two Hadamard matrices such that:

$$\mathbf{H}\mathbf{U}_k\mathbf{H} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi r_k} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+e^{i\pi r_k} & 1-e^{i\pi r_k} \\ 1-e^{i\pi r_k} & 1+e^{i\pi r_k} \end{bmatrix} \tag{41}$$

Creating his query machine! It is easy to show that:

$$\mathbf{H}\mathbf{U}_k\mathbf{H}\ |0\rangle = \frac{1}{2}\begin{bmatrix} 1+e^{i\pi r_k} \\ 1-e^{i\pi r_k} \end{bmatrix} = \frac{1+e^{i\pi r_k}}{2}|0\rangle + \frac{1-e^{i\pi r_k}}{2}|1\rangle \tag{42}$$

And therefore:

$$\mathbf{H}\mathbf{U}_k\mathbf{H}\ |0\rangle = |r_k\rangle \tag{43}$$

Then Bob inputs $|000\rangle$ to obtain:

$$(\mathbf{H}\mathbf{U}_k\mathbf{H} \otimes \mathbf{H}\mathbf{U}_k\mathbf{H} \otimes \mathbf{H}\mathbf{U}_k\mathbf{H})|000\rangle = |r_0\ r_1\ r_2\rangle \tag{44}$$

Therefore, after Bob's query, he knows the upside marks of the three cards! The set of upsides of the cards can be described by one of two three-qubit permutation sets:

$$\begin{aligned} S_0 &= \{|0\rangle, |0\rangle, |1\rangle\} \\ S_1 &= \{|0\rangle, |1\rangle, |1\rangle\} \end{aligned} \tag{45}$$

Or in other words, after Bob's query, he knows whether there are two cards diamond side up in the box, or whether there are two cards star side up in the box. Additionally, if the system is described by S_0, Bob knows that the winning card is one of the two cards with a diamond on top. This query machine has given Bob very important information about the game. So now Bob draws his card, only able to look at the top face. If the card Bob has drawn has a diamond on its upside face, he and Alice have an equal chance of winning, so he continues the game, however, if the card Bob has drawn has a star on its upside face, Bob knows he has drawn the losing card and refuses to continue playing. The opposite applies when the system is described by S_1. Thus, in allowing Bob to operate on the system with a quantum query machine and then decide whether to withdraw from the game, we have created a quantized version of this card game which can be guaranteed fair when Bob decides to continue playing.

Conclusion

In this paper, we only explored two relatively simple examples of quantum games. There are many more complicated games that range in their solutions. Some quantized classical games alter the odds, either skewing the game in one player's favor, or evening out a previously unfair game. For some classical games, quantization alters the game in its complexity but offers no notable change in the game's dynamic–a good example of this is perhaps the Stag Hunt game. We have only brushed the surface in our discussion, analyzing games which only require the fundamentals of quantum mechanics, however, as the games increase in their complexity, as do the quantum tactics needed to resolve them. Higher level quantum games often require the entanglement operator, extension to higher order bits, or in some cases, even cryptography to decode the answer.

While much work has been done in the field of quantum game theory since D. A. Meyer originated the term in 1999, there is still much to be done. As we continue to quantize new, more complicated games, adding to the already large set of games we can call quantum, perhaps we will find more tangible, real world applications. But for now, quantum games will continue to test our problem solving skills and our ability to apply the often intangible, strange realm of quantum to familiar ideas. Until the day when Alice and Bob can play tic-tac-toe on the world's first quantum computer, we will patiently wait, querying how quantum mechanics can upend our world–from classical mechanics to classical games.

Acknowledgements

We would like to acknowledge Umang Mehta, Paul Severino, and Harrison Price for their insightful comments on this paper, and of course, Savdeep Sethi for guidence throughout this quarter of Advanced Quantum.

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