Introduction to Quantum Error Correction

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March 23, 2020

1 Quantum Errors

In the proceeding, we will develop techniques for undoing unwanted transformations to a quantum system. We will call these unwanted transformations "quantum errors". First, it is useful to consider how classical errors differ from quantum ones. In classical hardware, such as the platter of a hard disk drive, the orientation of a local mangetic dipole moment in a ferromagnetic material is used to encode a binary bit, either 0 or 1. The magnetic dipole moment is produced by electrons in the material's atoms aligning their spin orientations and hence their intrisic magnetic dipole moments. This alignment is energetically favorable due to the "exchange energy" arising from Fermi-Dirac statistics. So, if an external magnetic field exerts a torque on a single electron's magnetic dipole moment sufficient to change its orientation relative to the ensemble, the electron will tend to realign its magnetic dipole moment with the ensemble. The situation is different in quantum hardware, where an experimentalist seeks to control the superposition of spin states of a single electron. There is no ensemble pressure for the individual electron to maintain its configuration in the presence of external noise. Further, in the classical case, the orientation of the material's electric dipole moment can only undergo discrete changes, say from 0 to 1. In the quantum case, we know that the spin of an individual electron exists in a superposition of the spin-up and spin-down states, which is discribed by a continuum.

Take the case of an isolated electron with the hamiltonian $H = \omega\frac{\sigma_z}{2} + u(t)\frac{\sigma_x}{2}$. This particle experiences a constant B-field in the z direction, generating a precession about the z-axis of the Bloch sphere at frequency ω. We may apply a B-field in the x direction at an amplitude $u(t)$ of our choosing. Experimentally, we may be controlling the B-field in the x direction by supplying Helmholtz coils with a current according to $u(t)$. However, we know that the thermal excitations of electrons in the cable that connects our function generator to our Helmholtz coils may lead to a B-field in the x direction behaving like $u(t) + \epsilon(t)$. Such an imperfection will induce errors in rotations about the x-axis of the Bloch sphere. For example we may intend to rotate by π radians about the x-axis $|0\rangle \to |1\rangle$ but instead rotate by $\pi + \epsilon$ radians about the x-axis $|0\rangle \to -i\sin\left(\frac{\epsilon}{2}\right)|0\rangle + \cos\left(\frac{\epsilon}{2}\right)|1\rangle$. It would be convenient if we could describe any quantum error by a unitary operator E acting on our state $|\psi\rangle$.

However, it is not obvious that we can always describe the evolution of our system by a unitary operation. In cases where our system exhibits dissipation, it loses energy, hence information, to the environment. This is equivalent to saying that the transformation does not preserve trace, and is not unitary. An example that the reader may have encountered in an undergraduate experimental physics sequence is the decay of a neutron into a proton, electron, and electron neutrino $n \to p + e + \nu_e$. Say that at some time t_0 we always detect a neutron. This decay follows poisson statistics so we know that at some later time t we will detect a neutron with probability $e^{\frac{-(t-t_0)}{\tau}}$. In the density matrix formalism, we write

$\mathrm{Tr}(\rho_n(t)) = e^{\frac{-(t-t_0)}{\tau}}$ to indicate that finding a neutron in any state is equal to the given probability. However, we know that the probability of finding a proton, electron, and electron neutrino is inversely related to that of finding a neutron, $\mathrm{Tr}(\rho_p(t)\otimes\rho_e(t)\otimes\rho_\nu(t)) = 1-e^{\frac{-(t-t_0)}{\tau}}$. Thus, $\mathrm{Tr}(\rho_n\otimes\rho_p\otimes\rho_e\otimes\rho_\nu) = 1$ for all t. So we see that, although the evolution of our principal system ρ_n is not trace-preserving, the evolution of the principal system and the channels it decays into is trace preserving.

The quantum operation formalism is useful for describing evolutions of this type. We start by considering a principal system on a Hilbert space $\mathcal{H}_a$ coupled to an environment on a Hilbert space $\mathcal{H}_b$. We write $\rho_{ab} = \rho_a \otimes \rho_b$. We can evolve the joint system under a unitary operation $\rho_{ab}^{'} = U\rho_{ab}U^\dagger$. We may put our environment in some initial state $|b_0\rangle$ then $\rho_{ab} = \rho_a \otimes |b_0\rangle\langle b_0|$. We have

$$\begin{aligned}\rho_a^{'} &= \mathrm{Tr}_b(\rho_{ab}^{'}) \\ &= \sum_k \mathbb{1}\otimes\langle b_k|\,(U(\rho_a\otimes|b_0\rangle\langle b_0|)U^\dagger)\mathbb{1}\otimes|b_k\rangle \\ &= \sum_k E_k\rho_a E_k^\dagger\end{aligned}$$

$\{E_k\}$ are sometimes referred to as Kraus operators. $\{E_k\}$ are not necessarily unitary. This representation is useful because we can consider how the principal system will evolve without having to consider the many degrees of freedom of our environment. We will only need d^2 linearly independent Kraus operators where $d = \dim(\mathcal{H}_a)$ because that is sufficient to represent any action on ρ_a.

In analogy to the measurement operator formalism, one may obtain [8]

$$\rho_a^k = \frac{E_k\rho_a E_k^\dagger}{\mathrm{Tr}(E_k\rho_a E_k^\dagger)} \tag{1}$$

which describes the state after the k$^{\text{th}}$ outcome has occurred. The k$^{\text{th}}$ outcome will occur with probability $\mathrm{Tr}(E_k\rho_a E_k^\dagger)$.

We consider the example of amplitude damping. This situation arises when, for instance, an excited electron spontaneously emits a photon. Let $|0\rangle$ be the electron ground state and $|1\rangle$ be the excited state. Suppose that the electron decays to the ground state with probability p at every instant. If the electron is initially in the ground state, it will stay in the ground state with probability 1. If the electron is initially in the excited state, it will stay in the excited state with probability $1-p$ or decay to the ground state with probability p. The first Kraus operator E_0 represents the "stay" trajectory and the second Kraus operator E_1 represents the "decay" trajectory.

$$E_0 = \begin{pmatrix}1 & 0\\ 0 & \sqrt{1-p}\end{pmatrix}\ E_1 = \begin{pmatrix}0 & \sqrt{p}\\ 0 & 0\end{pmatrix}$$

If we put the electron in the excited state $\rho = |1\rangle\langle 1|$ we find

$$\begin{aligned}\rho^{'} &= E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger \\ &= p\,|0\rangle\langle 0| + (1-p)\,|1\rangle\langle 1|\end{aligned}$$

Notice that our system began in a pure state $\mathrm{Tr}(\rho^2) = 1$ and evolved into a mixed state $\mathrm{Tr}(\rho^{'2}) = 1 - 2p + 2p^2$. For example, take $p = \frac{1}{2}$ then $\mathrm{Tr}(\rho^{'2}) = \frac{1}{2} < 1$. Thus, the quantum error correction schemes we develop must be able to recover a pure state from the possibly mixed state produced by an error.

To perform quantum error correction, we will entangle an ancillary system with our principal system, and upon projective measurement of the ancillary system, the mixed state of the principal system will be projected onto a pure state. Furthermore, we will be able to uniquely determine the operation required to correct the pure state of the principal system to the origninal one. Write the principal system $|\psi\rangle$, and the ancillary system $|\phi\rangle$. A general quantum error takes the form

$$|\psi\rangle \otimes |\phi_0\rangle \to \{(E_i\,|\psi\rangle) \otimes |\phi_i\rangle\}$$

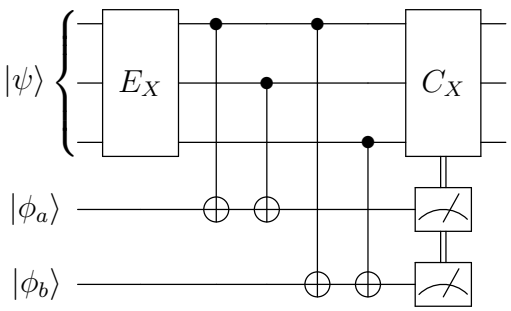
If the $\{\phi_i\}$ are unique, measuring the ancillary system with an outcome ϕ_i will determine that the principal system is in the state $E_i\,|\psi\rangle$, and we will apply the operation E_i^{-1} to correct the principal system. We will call the measurement ϕ_i the syndrome.

Furthermore, we can decompose each continuous error E_i, an arbitrary operator on a Hilbert space, into a basis of discrete errors. First, note that any 2×2 operator can be represented by a complex linear combination of the Pauli matrices with the identity $\{I, X, Y, Z\}$. Then note that $Y = iXZ$, so we may equivalently choose the basis $\{I, X, Z, XZ\}$. So, we decompose

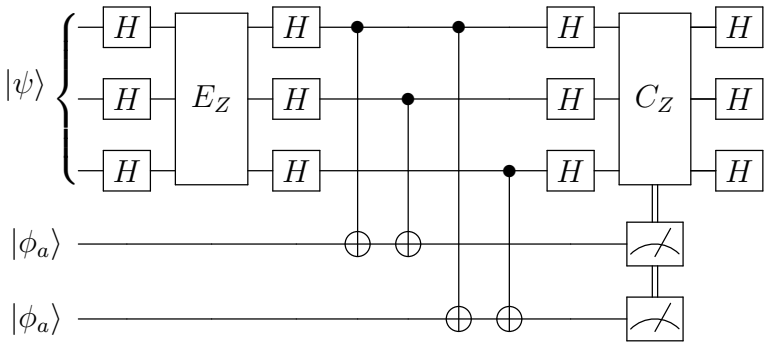
$$\begin{aligned}(E_i\,|\psi\rangle) \otimes |\phi_i\rangle \to\; & |\psi\rangle \otimes \left|\phi_i^I\right\rangle \\ & + X\,|\psi\rangle \otimes \left|\phi_i^X\right\rangle \\ & + Z\,|\psi\rangle \otimes \left|\phi_i^Z\right\rangle \\ & + XZ\,|\psi\rangle \otimes \left|\phi_i^{XZ}\right\rangle\end{aligned}$$

where our error correction scheme must guarantee that the $\{\left|\phi_i^A\right\rangle\}$ are unique for $A \in \{I, X, Z, XZ\}$. This decomposition generalizes to operatons on n qubits by choosing the basis $\{I, X, Z, XZ\}^{\otimes n}$. Analogous bases can be chosen for qudits [6].

Before we introduce some schemes for quantum error correction, it is worth introducing the quantum treshold theorem. This theorem states, in brief, that a quantum computer with a constant probability of error p associated with each gate can execute a quantum program with arbitrary precision in polynomial time given that p is below a fixed threshold [1]. Without this theorem, it is not obvious that one could perform error correction with noisy gates and hope to achieve better performance. For instance, it could have been the case that quantum mechanics prohibited us from using noisy quantum gates to create near-perfect quantum gates. A quantum computer that is able to achieve arbitrary accuracy is said to be fault-tolerant. Estimates for the upper bound on p vary, and are determined by the efficiency of the error correction scheme employed.



(a) X error, syndorme measurement, and correction.



(b) Z error, syndorme measurement, and correction.

Figure 1: Error (E), syndrome measurement, and correction (C) circuits for the 3-qubit repetition code. Note that we initialize $|\phi_a\rangle = |\phi_b\rangle = |0\rangle$. The input state is allowed to be an arbitrary linear combination of the basis states $|\psi\rangle = \alpha\,|\bar{0}\rangle + \beta\,|\bar{1}\rangle$.

2 Repetition Codes

In the previous section we saw that we may decompose any quantum error in the basis $\{I, X, Z, XZ\}$. This basis has a natural interpretation with I representing no error, X a bit-flip error, Z a phase-flip error, and XZ a bit- and phase-flip error. It turns out that we can correct both bit-flip and phase-flip errors, individually, by drawing on a classical error correction code.

In the classical 3-bit repitition code, we define a logical basis $\bar{0} = 000$ and $\bar{1} = 111$. We say that each logical bit $\bar{0}$, $\bar{1}$ consists of three physical bits 000, 111. When we act on each logical bit with a gate, or radiation is incident on our classical hardware, there is a probability that an error takes a logical bit out of the logical basis. For example, a logical bit may suffer a bit-flip on one of its physical bits $111 \to 101$. In the classical case we can directly measure our state 101. We see that 101 differs by only one bit from one of our logical basis elements 111, and correct the state $101 \to 111$. However, this scheme fails if a logical bit suffers two bit-flips $111 \to 100$. In this case, 100 looks more like the logical basis element $\bar{0} = 000$ than $\bar{1} = 111$. It is more likely that one physical bit flipped than two, so our error correction mechanism would mistakenly change $100 \to 000$. However, we can see that repeating more physical bits $\bar{0} = 0^d$ and $\bar{1} = 1^d$ will increase the number of bit-flips that each logical bit may suffer before it will become confused with the other logical basis element. In the case where we use d physical bits per logical bit, we can correct up to $\lfloor \frac{d-1}{2} \rfloor$ errors. In general, we will say a code of distance d that uses n physical bits to encode k logical bits is an $[n, k, d]$ code. The 3-bit repetition code is a $[3, 1, 3]$ code.

In the quantum analogue of the 3-bit repetition code we define $|\bar{0}\rangle = |000\rangle$ and $|\bar{1}\rangle = |111\rangle$. We will write X_i to denote the operator that acts with X on the i^{th} qubit where we supress the tensor products $\bigotimes_{k=1}^{i-1} I \otimes X \otimes \bigotimes_{k=i+1}^{n} I$. Consider a state in the logical basis $|\psi\rangle = |\bar{0}\rangle$ input to the circuit shown in Figure 1a. Suppose this state suffers a bit-flip error on its first physical bit $X_1\,|000\rangle = |100\rangle$. The first ancilla qubit will be flipped as a result of the CNOT with the first physical qubit $|\phi_a\rangle \to |1\rangle$. Similarly, the second ancilla will be flipped as a

result of the CNOT with the first physical qubit $|\phi_b\rangle \to |1\rangle$. We call the measurement of the ancilla qubits $\phi_a\phi_b = 11$ the syndrome. The syndrome 11 is unique for the error X_1. Thus, we can apply X_1 to $|\psi\rangle$ to correct the state. Intuitively, the measurement of ϕ_a will be 1 if the first and second qubits differ in parity, while ϕ_b will be 1 if the first and third qubits differ in parity. The Z syndrome measurement and correction circuit shown in Figure 1b is analogous to that for X. In the X basis given by $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, a phase-flip looks like a bit-flip $Z|+\rangle = |-\rangle$.

$\phi_a\phi_b$ syndrome	C_X correction	C_Z correction
00	I	I
01	X_3	Z_3
10	X_2	Z_2
11	X_1	Z_1

Table 1: Syndrome measurements and corrections for the 3-qubit repetition code circuits.

Consider the case where we have a linear combination of errors $E = \frac{1}{\sqrt{2}}(X + I)$. Suppose the initial state is incident on the X error correction circuit $|\psi\rangle = |\bar{0}\rangle$. After applying E, we have $|\psi\rangle = \frac{1}{\sqrt{2}}(|100\rangle + |000\rangle)$. After applying the CNOTs to the first ancilla, we have $|\psi;\phi_a\rangle = \frac{1}{\sqrt{2}}(|1001\rangle + |0000\rangle)$. Thus, if measurement yields $\phi_a = 1$, we know $|\psi\rangle = |100\rangle$ and if $\phi_a = 0$ then $|\psi\rangle = |000\rangle$. We see that the projective measurement of the ancilla projects the linear combination of errors on to a single one, and the correct syndrome is determined. We have considered examples where $|\psi\rangle = |\bar{0}\rangle$ for simplicity, but all gates are linear so a linear combination of the logical basis $|\psi\rangle = \alpha|\bar{0}\rangle + \beta|\bar{1}\rangle$ may be corrected.

So, we have examined an error correction scheme that may correct arbitrary linear combinations of I and X or I and Z errors. In the next section, we will develop a formalism for describing error correcting codes. This will allow us to make garauntees about what errors the code can and cannot correct, as well as how to define logical operations on the code space.

3 Stabilizer Formalism

Observe that, in the case of measuring an X error syndrome in the repetition code, we first compared the parity of the first and second qubit, and then the parity of the first and third qubit. This step is equivalent to measuring the eigenvalues of the operators Z_1Z_2 and Z_1Z_3. Concretely, suppose $|\bar{0}\rangle$ experiences a bit flip error on the first qubit X_1. We have $Z_1Z_2X_1|000\rangle = Z_1Z_2|100\rangle = -|100\rangle$. Similarly, $Z_1Z_3X_1|000\rangle = Z_1Z_3|100\rangle = -|100\rangle$. The eigenvalue corresponding to the erroneous state $X_1|\psi\rangle$ is -1 for Z_1Z_2 and -1 for Z_1Z_3. This measurement is analogous to the syndrome we computed for the repetition code, except we replace a measurement of 1 or 0 on the ancillas with measurements corresponding to the eigenvalues -1 or 1. We will say that Z_1Z_2 and Z_1Z_3 are stabilizers of our code because they fix our code words $|\bar{0}\rangle$ and $|\bar{1}\rangle$, i.e. our code words are eigenvectors of the stabilizers

with eigenvalues $+1$. The stabilizer formalism will allow us to talk about stabilizers on a code space, without having to directly refer to the code words themselves.

Take $\mathcal{H}_n$ as a Hilbert space over $\mathbb{C}^{2^n}$ equipped with the standard Hermitian inner-product. Let $\hat{\mathcal{H}}_n$ be the space of operators on $\mathcal{H}_n$. The Pauli group $\mathcal{P}_n = \{\pm 1, \pm i\} \times \{I, X, Y, Z\}^{\otimes n}$ forms a group under multiplication. Again, we can write any matrix $E \in \hat{\mathcal{H}}_n$ as a complex linear combination of the generators of $\mathcal{P}_n$, and therfore any error on n qubits. A stabilizer is an Abelian subgroup of the Pauli group $\mathcal{S} \subset \mathcal{P}_n$ with $-I \notin \mathcal{S}$. We can take a linearly independent set of m generators for $\mathcal{S}$ to be

$$\mathcal{M} = \{M_i : 1 \leq i \leq m\} \subset \mathcal{P}_n$$

We define the code space, i.e. the logical basis states

$$D = \{|\psi\rangle \in \mathcal{H}_n : \ \forall \ M \in \mathcal{M}, M\,|\psi\rangle = |\psi\rangle\}$$

We say that $\mathcal{S}$ stabilizes D.

Here is the magic. We will use the $+1$ or -1 eigenvalue measurements of the stabilizers $M \in \mathcal{M}$ to determine the syndrome of an error E. Observe that two elements in the Pauli group $G, E \in \mathcal{P}_n$ either commute $EG = GE$ or anti-commute $EG = -GE$. To see this, first observe that each pair formed from I, X, Y, and Z either commute or anti-commute. Then consider $E, G \in \mathcal{P}_n$. At every position i in the tensor product expansion of E or G we will have an operator I, X, Y, or Z. Then, either $E_iG_i = G_iE_i$ or $E_iG_i = -E_iG_i$. The commutator $[E, G]$ will accrue all factors of 1 or -1 over the n positions. Suppose for simplicity that an error $E \in \mathcal{P}_n$ acts on a code word. Our errors most generally are operators in $\hat{\mathcal{H}}_n$, but as shown in section 1, we may consider only generators of the Pauli group. Consider $M \in \mathcal{M}$ and $|\psi\rangle \in D$. $M\,|\psi\rangle = |\psi\rangle \ \forall \ M \in \mathcal{M}$. Thus, a valid code word will result in a measurement of the eigenvalue $+1$ for all stabilizers. If $ME = -EM$ then $ME\,|\psi\rangle = -EM\,|\psi\rangle = -E\,|\psi\rangle$. We say M detects the error E and we will use the measurement of the -1 eigenvalue for M as part of the syndrome of E. We define the syndrome $f_M : \mathcal{P}_n \to \mathbb{Z}_2$

$$f_M(E) = \begin{cases} 0, \ [M, E] = 0 \\ 1, \ \{M, E\} = 0 \end{cases}$$

Thus, a full syndrome for E is constructed by concatening the values of $f_M(E)$ for all $M \in \mathcal{M}$. Note that because we use the stabilizers $M \in \mathcal{M}$ for syndrome measurement we must have $[M, M'] = 0 \ \forall \ M, M' \in \mathcal{M}$, hence the requirement that $\mathcal{S}$ be Abelian. Further, we can not have $-I \in \mathcal{S}$ otherwise $-I\,|\psi\rangle = |\psi\rangle \implies |\psi\rangle = 0$ and hence D is trivial.

We have not yet considered the case where $[M, E] = 0 \ \forall \ M \in \mathcal{M}$. If $E \notin \mathcal{M}$ then E will produce the all-zero syndrome $0 \ldots 0$ and is an "undetected" error. These operations move our state within the code space, and we will see that we can use some of them to define logical operations on the code words. We define the centralizer of $\mathcal{S}$

$$\mathcal{C}(\mathcal{S}) = \{G \in \mathcal{P}_n : \ \forall \ S \in \mathcal{S}, GS = SG\}$$

We define the normalizer of $\mathcal{S}$

$$\mathcal{N}(\mathcal{S}) = \{G \in \mathcal{P}_n : \ \forall \ S \in \mathcal{S}, \ \exists \ T \in \mathcal{S} \text{ s.t. } GS = TG\}$$

Since $\mathcal{S}$ is Abelian, we have $\mathcal{C}(\mathcal{S}) = \mathcal{N}(\mathcal{S})$. All of the logical operations on our code live in $\mathcal{N}(\mathcal{S}) - \mathcal{S}$. If $E \in \mathcal{N}(\mathcal{S}) - \mathcal{S}$ then $ES = SE$ so $SE|\psi\rangle = ES|\psi\rangle = E|\psi\rangle$. But in general $E|\psi\rangle \neq |\psi\rangle$. It can be shown that there exists an automorphism $\mathcal{N}(\mathcal{S}) - \mathcal{S} \to \mathcal{P}_k$ [6]. In particular, a subset of the generators of $\mathcal{N}(\mathcal{S}) - \mathcal{S}$ are equivalence classes $\bar{X}_i$ and $\bar{Z}_i$ $i = 1, \ldots k$ which map to X_i and Z_i in $\mathcal{P}_k$, respectively. These operations give us a natural way to apply logical operations to our code words. We have $m = n - k$. So we see that a code with m stabilizers on n phyiscal qubits will give us a k logical qubit code.

One might intuitively think that all correctable errors are those $E \notin \mathcal{N}(\mathcal{S})$ because they produce a non-zero syndrome. However, in order for a set of errors to be correctible, they must satisfy more stringent requirements [9]. In particular, two errors E_a, E_b on orthogonal code words $|\psi_i\rangle, |\psi_j\rangle$ must be distinguishable, which gives the requirement

$$\langle\psi_j| E_b^\dagger E_a |\psi_i\rangle = 0$$

If the erroneous states were not orthogonal, they would produce the same syndrome. Additionally, the syndrome of an error must not give us information about the code word, otherwise we have learned information about the quantum state that we wish to encode, which gives the requirement

$$\langle\psi_j| E_b^\dagger E_a |\psi_j\rangle = \langle\psi_i| E_b^\dagger E_a |\psi_i\rangle$$

These two conditions are frequently combined

$$\langle\psi_j| E_b^\dagger E_a |\psi_i\rangle = C_{ab}\delta_{ij}$$

Additionally, there exists a relationship that allows us to determine if there exists a code on n physical qubits to encode k logical qubits and correct up to t single qubit errors. This relationship is known as the quantum hamming bound, in analogy to the classical hamming bound [5].

$$2^k \sum_{l=0}^{t} 3^l \binom{n}{l} \leq 2^n$$

Intuitively, this formula tells us that there must be a unique representation in n physical qubits for all possible combinations of t errors that can occur on k logical qubits. The factor of 3 comes from the fact that we can expand any non-trivial error in X, Z, and XZ. This relationship only holds for non-degenerate codes. Define the weight of an operator $E \in \mathcal{P}_n$ as the number of non-identity terms that appear in its tensor product expansion. We say the minimum distance of a stabilizer code is

$$d_{min} = \min_{E \in \mathcal{N}(\mathcal{S}) - \mathcal{S}} \text{weight}(E)$$

We say that $\mathcal{S}$ forms a degenerate stabilizer code if $\mathcal{S}$ contains an element of weight less than d_{min}. This is equivalent to the condition that C_{ab} is singular.

4 Surface Codes

We now give an example of one of the most important stabilizer codes. The surface codes are a family of stabilizer codes where physical qubits are placed on a 2D lattice with nearest neighbor connectivity (2DNN). Perhaps the most famous surface code is the toric code introduced by Kitaev [7]. The toric code employs periodic boundary conditions on the 2D lattice consistent with a genus 1 surface, e.g. a torus. Constructing a Hamiltonian to realize a system with properties that exhibit the structure of this code is an active area of research in the field of topological quantum computing, and we refer the interested reader to Kitaev's paper. Here, we take up the planar surface code. The planar surface code is of practical interest and can be deployed on an arbitrary quantum architecture.

We begin with a 2D lattice where a physical qubit is placed on each edge connecting two vertices, see Figure 2. The planar surface code is defined by the stabilizers $\mathcal{M} = \{A_v\} \cup \{B_p\}$ where

$$A_v = \prod_{e \in v} X_e \qquad B_p = \prod_{e \in p} Z_e$$

We will say A_v is an X stabilizer and B_p is a Z stabilizer. e represents the index of a qubit associated with an edge. v represents a vertex. p represents a plaquette–a surface lying inside of three or more edges. We create a stabilizer for each vertex and plaquette in the lattice. In the planar surface code, we also remove all of the outermost edges from two non-adjcent boundaries of the lattice, forming smooth and rough boundaries. Smooth boundaries have X stabilizers with 3 physical qubits and Z stabilizers with 4 physical qubits. Rough boundaries have X stabilizers with 4 physical qubits and Z stabilizers with 3 physical qubits.

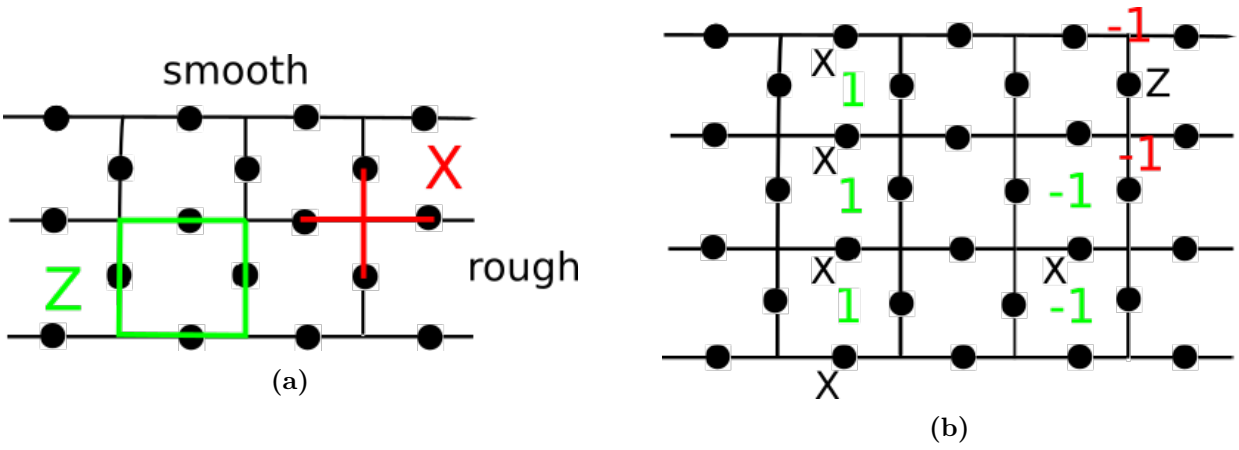


Figure 2: a) Planar surface code. Physical qubits are represented by black circles. Examples of X and Z stabilizers are highlighted. X stabilizers are formed by 3 or 4 qubits on a vertex. Z stabilizers are formed by 3 or 4 qubits on a plaquette. Rough and smooth boundaries are annotated. Ancilla qubits used to measure stabilizer eigenvalues are not pictured. b) Examples of X and Z errors on a planar surface code. The X error chain on the left is undetected by the Z stabilizers. The X error near the bottom will produce -1 eigenvalue measurements for the adjacent Z stabilizers. Similarly, the Z error near the top will produce -1 eigenvalue measurements for the adjacent X stabilizers.

We say that a clean surface is one where all stabilizers fix the state of the qubits they

act on. Equivalently, valid code words are eigenvectors with eigenvalue $+1$ for all stabilizers. We will employ the X stabilizers to detect Z errors and Z stabilizers to detect X errors. If qubit i is affected by a Z error, written Z_i, the X stabilizers that contain an X_i term will anticommute with Z_i, and the erroneous state will be an eigenvector of those stabilizers with an eigenvalue -1. We may measure this eigenvalue using an ancilla qubit, determining the syndrome of an error. When a single stabilizer experiences two errors of the same type, the errors will commute with the stabilizer, yielding an eigenvalue measurement of $+1$ for the erroneous state. In cases where many errors occur in sequence, this can produce undetected errors, see Figure 2b. Thus, the number of single qubit errors that the planar code can tolerate is proportional to the minimum distance between any two non-adjacent boundaries called the length L. The number of physical qubits required to increase the size of the lattice is quadratic in the length, but the probability of an uncorrectable error occuring is exponentially decreasing in the length $\propto p^L$ with $p \leq 1$.

Furthermore, the planar surface code has the nice property that each physical qubit is involved in at most 4 stabilizers and each stabilizer acts on at most 4 physical qubits. Although physical proximity is not required to dutifully execute the surface code protocol, the locality of operations makes it easier to experimentally realize [2]. Note also that the planar surface code can correct arbitrary errors, namely X, Z, and XZ, since an XZ error will be corrected by X and Z stabilizers independently.

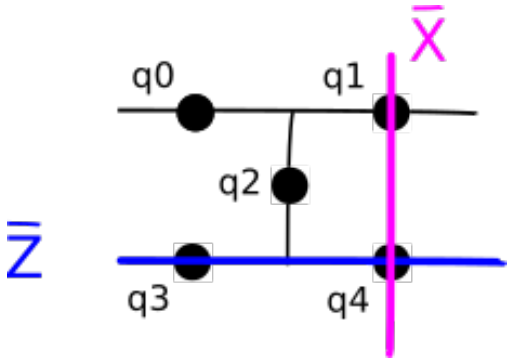


Figure 3: Minimum viable example of a planar surface code.

We take up the example of the minimum viable planar surface code to illustrate connections with the stabilizer formalism developed in the previous section, see Figure 3. The stabilizers for this code are generated by

$$\mathcal{M} = \{X_0X_1X_2, X_2X_3X_4, Z_0Z_2Z_3, Z_1Z_2Z_4\}$$

Observe that $[M, M^{'}] = 0 \; \forall \; M, M^{'} \in \mathcal{M}$. By applying all possible combinations of stabilizers to the state $|00000\rangle$ we find

$$|\bar{0}\rangle = |00000\rangle + |11100\rangle + |00111\rangle + |11011\rangle$$

Define

$$\bar{X} = X_1X_4 \quad \bar{Z} = Z_3Z_4$$

Observe that $\bar{X}, \bar{Z} \in \mathcal{N}(\mathcal{S}) - \mathcal{S}$. Our choices could have been $\bar{X} = X_0X_3$ or $\bar{Z} = Z_0Z_1$. What is important is that each of these operations use two physical qubits on the same stabilizer, and hence commute with that stabilizer. We have

$$|\bar{1}\rangle = \bar{X}\,|\bar{0}\rangle = |1010\rangle + |01110\rangle + |01001\rangle + |10010\rangle$$

Note that $\forall\ M \in \mathcal{M},\ M\,|\bar{1}\rangle = |\bar{1}\rangle$. Also, $\bar{Z}\,|\bar{1}\rangle = -\,|\bar{1}\rangle$.

As we extend the planar surface code to lengths greater than our minimum viable example, the code will be able to tolerate more errors, but will still encode one logical qubit. Further, we have not described a way to realize a univeral gate set using this code, or construct interactions between two logical qubits. However, the same principles demonstrated here are applicable to codes that can accomplish those tasks, which the reader should now be able to attack [3] [4].

5 Acknowledgements

This work was submitted in partial fulfillment of the final project requirement for PHYS 24300/1. The author thanks Srivatsan Chakram for helpful conversations and insight. Quantum circuits were typeset using qcircuit. Graphics were produced in inkscape, an open-source vector graphics tool.

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