Problem Set 3
Physics 341
Due October 21

Some abbreviations: S - Shankar.

1. These questions came up in lecture.

(i) Imagine you have an operator \(T : V \to V\) where \(V\) is an \(n\)-dimensional vector space. In some basis \(\{|e_i\}\) this operator has a matrix presentation \(M\). Suppose you change to a new basis \(|e'_i\rangle = S_{ij}|e_j\rangle\). What must be true about the matrix \(S\)? In this primed basis, derive the matrix representation of \(T\) in terms of \(M\) and \(S\).

(ii) The Dirac delta-function has many nice properties. Show that \(\delta(ax) = \frac{\delta(x)}{|a|}\). Show that the limit
\[
\delta(x) = \lim_{a \to 0} \frac{1}{\pi x^2 + a^2}
\]
satisfies the properties of the Dirac delta-function. Namely, that \(\delta(x)\) vanishes if \(x \neq 0\) and
\[
\int_{-\infty}^{+\infty} \delta(x) = 1.
\]

2. Let’s continue with a straightforward warm-up exercise. Consider four operators \(A, B, C, D\). Show that
\[
\]

Now consider \(e^A = e^{\lambda B} e^{\lambda C}\) where \(A, B, C\) are operators and \(\lambda\) is a small parameter. Find an expression for \(A\) in terms of \(B\) and \(C\) including terms of \(O(\lambda^3)\). Evaluate this expression for \(B = x\) and \(C = p\).

3. Consider a wavefunction that behaves as follows:
\[
\psi(x) = \begin{cases} 
0 & \text{if } x < 0, \\
2 & \text{if } 0 \leq x \leq x', \\
0 & \text{if } x > x'.
\end{cases}
\]

where \(x' > 0\) is fixed and \(x\) runs from \((-\infty, \infty)\).

(i) What is the probability density that the particle can be found at \(x = x'/2\)?

(ii) Please find the momentum space wavefunction, \(\psi(p)\), for this particle.

(iii) What is the momentum space wavefunction for \(\partial_x \psi(x)\)?

(iv) Can you express \(\partial_x \psi(x)\) in terms of delta-functions?
(v) Finally, let us think about $\psi(x)$ now as a wavefunction defined on the interval $[-\pi, \pi]$ so both

$$-\pi \leq x \leq \pi, \quad 0 < x' < \pi.$$ 

What is the Fourier series representation of $\psi(x)$?

4. A particle of mass $m$ moves in one dimension in an infinite square well potential extending from $-a$ to $+a$. At time $t = 0$, the system is described by the wavefunction

$$\Psi(x, 0) = c (\Psi_0(x) + 3\Psi_1(x))$$

where $\Psi_0$ and $\Psi_1$ are the normalized eigenfunctions for the ground state and first excited state, respectively.

(i) Find the value of $c$ for which the wavefunction is normalized.

(ii) Compute the probability that the particle is found in the interval $-a < x < 0$ at time $t$.

5. Consider a particle in a box with potential $V(x) = \infty$ for $x < 0$ and $x > a$. Let $V = 0$ for $0 < x < a$. Determine the energy levels and normalized energy eigenfunctions. You will find that the energy eigenstates are labeled by an integer $n$. Let $|n\rangle$ denote the corresponding energy eigenstate. Compute the expectation values

$$\langle x \rangle = \langle n|x|n\rangle, \quad \langle (\Delta x)^2 \rangle = \langle n|(x - \langle x \rangle)^2|n\rangle$$

and show that in the limit of large $n$, these quantum expectation values become equal to the average values for classical motion of a particle in a box. The classical averages are computed as averages over the classical motion.