Solutions to Problem Set 3

Physics 342

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1 Gates

In the \(\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}\) basis, the Hadamard matrix is

\[
H = \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix} \otimes \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}
\]

and the matrix for the two qubit CNOT gate is

\[
\text{CNOT} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

So the effective matrix for the circuit is

\[
H \cdot \text{CNOT} \cdot H = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix} \cdot \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} \cdot \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

which is exactly the matrix for the circuit on the right. This can be verified by applying the matrix on a general state \(a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle\), which gives \(a|00\rangle + b|11\rangle + c|10\rangle + d|01\rangle\) which is what we expect when the second qubit acts as a control bit.
2 More gates

(i) The toffoli gate inverts only if the control bits are both 1. Classical truth table:

<table>
<thead>
<tr>
<th>a b c</th>
<th>a b c ⊕ ab</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 0 0</td>
<td>0 0 0</td>
</tr>
<tr>
<td>0 0 1</td>
<td>0 0 1</td>
</tr>
<tr>
<td>0 1 0</td>
<td>0 1 0</td>
</tr>
<tr>
<td>0 1 1</td>
<td>0 1 1</td>
</tr>
<tr>
<td>1 0 0</td>
<td>1 0 0</td>
</tr>
<tr>
<td>1 0 1</td>
<td>1 0 1</td>
</tr>
<tr>
<td>1 1 0</td>
<td>1 1 1</td>
</tr>
<tr>
<td>1 1 1</td>
<td>1 1 0</td>
</tr>
</tbody>
</table>

(ii) Let us write the matrices for each of the steps in the circuit for the basis \{\ket{000}, \ket{001}, \ket{010}, \ket{011}, \ket{100}, \ket{101}, \ket{110}, \ket{111}\}.
Equivalent matrix $M_4M_2M_3M_2M_1 =$

$$
\begin{pmatrix}
\mathbb{I}_2 & \mathbb{I}_2 \\
\mathbb{I}_2 & \mathbb{I}_2 \\
\mathbb{I}_2 & \mathbb{I}_2 \\
\mathbb{I}_2 & \mathbb{I}_2
\end{pmatrix}
\cdot
\begin{pmatrix}
\mathbb{I}_2 & 0 \\
0 & \mathbb{I}_2 \\
\mathbb{I}_2 & 0 \\
\mathbb{I}_2 & 0
\end{pmatrix}
\cdot
\begin{pmatrix}
\mathbb{I}_2 \\
0 \\
\mathbb{I}_2 \\
\mathbb{I}_2
\end{pmatrix}
\cdot
\begin{pmatrix}
\mathbb{I}_2 & 0 \\
0 & \mathbb{I}_2 \\
\mathbb{I}_2 & 0 \\
\mathbb{I}_2 & 0
\end{pmatrix}
\cdot
\begin{pmatrix}
\mathbb{I}_2 \\
\mathbb{I}_2 \\
\mathbb{I}_2 \\
\mathbb{I}_2
\end{pmatrix}
$$

$$
\begin{pmatrix}
\mathbb{I}_2 \\
\mathbb{I}_2 \\
\mathbb{I}_2 \\
\mathbb{I}_2
\end{pmatrix}
\cdot
\begin{pmatrix}
\mathbb{I}_2 \\
\mathbb{I}_2 \\
\mathbb{I}_2 \\
\mathbb{I}_2
\end{pmatrix}
\cdot
\begin{pmatrix}
\mathbb{I}_2 \\
\mathbb{I}_2 \\
\mathbb{I}_2 \\
\mathbb{I}_2
\end{pmatrix}
\cdot
\begin{pmatrix}
\mathbb{I}_2 \\
\mathbb{I}_2 \\
\mathbb{I}_2 \\
\mathbb{I}_2
\end{pmatrix}
\cdot
\begin{pmatrix}
\mathbb{I}_2 \\
\mathbb{I}_2 \\
\mathbb{I}_2 \\
\mathbb{I}_2
\end{pmatrix}
$$

This is equivalent to the circuit shown below with the first two qubits as control qubits and the gate $V^2$ acting on the third qubit.
3 3-qubit GHZ state

The following circuit acting on the state $|000\rangle$ gives us the required state:

Let’s confirm:

$$H_1|000\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |100\rangle) = |\psi_1\rangle$$

$$\text{CNOT}_{12}|\psi_1\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |110\rangle) = |\psi_2\rangle$$

$$\text{CNOT}_{23}|\psi_2\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

which is what we wanted. These states are called GHZ states after Greenberger–Horne–Zeilinger and are maximally entangled states.
4 Density matrices

(i) Let us take two density operators $\rho_1$ and $\rho_2$. For the statement to be true, any point on the ‘line segment’ joining the two operators should also be a density operator, that is, it should satisfy three properties:

- It should be self-adjoint (since we are working with a finite-dimensional Hilbert space, it is the same as being a Hermitian operator)

$$\rho^\dagger = (\lambda \rho_1 + (1 - \lambda)\rho_2)^\dagger = \lambda \rho_1^\dagger + (1 - \lambda)\rho_2^\dagger = \rho$$

- Its trace should be 1

$$\text{Tr} (\rho) = \text{Tr} (\lambda \rho_1 + (1 - \lambda)\rho_2)$$
$$= \text{Tr} (\lambda \rho_1) + \text{Tr} ((1 - \lambda)\rho_2)$$
$$= \lambda \text{Tr} (\rho_1) + (1 - \lambda) \text{Tr} (\rho_2)$$
$$= \lambda + (1 - \lambda)$$
$$= 1$$

- The operator should be positive (recall definition from HW1!). For an arbitrary state $|\psi\rangle$

$$\langle \psi | \rho | \psi \rangle = \lambda \langle \psi | \rho_1 | \psi \rangle + (1 - \lambda) \langle \psi | \rho_2 | \psi \rangle$$

Everything on the right hand side is greater than or equal to 0 because $\rho_1$ and $\rho_2$ are positive operators, thus $\langle \psi | \rho | \psi \rangle \geq 0$.

Thus, density operators do indeed form a convex set.

(ii) Suppose the density operator for a pure state, $\rho$, could be written as a linear combination of two other density operators in a form given in the part (i):

$$\rho = \lambda \rho_1 + (1 - \lambda)\rho_2$$

Let $|\psi_\perp\rangle$ be a vector orthogonal to $|\psi\rangle$. Then,

$$\langle \psi_\perp | \rho | \psi_\perp \rangle = 0$$
$$= \lambda \langle \psi_\perp | \rho_1 | \psi_\perp \rangle + (1 - \lambda) \langle \psi_\perp | \rho_2 | \psi_\perp \rangle$$

The last expression can only vanish if $\lambda$ is either 0 or 1 (in which case $\rho$ not a sum of two operators anymore), or if both $\langle \psi_\perp | \rho_1 | \psi_\perp \rangle$ and $\langle \psi_\perp | \rho_2 | \psi_\perp \rangle$ vanish for any $|\psi_\perp\rangle$, which is not possible unless $\rho_1 = \rho_2 = \rho$. 


(iii)
Only pure states are extremal points, because for any mixed state it is possible to diagonalize the matrix to make it of the form $\text{Diag}(p_1, p_2...)$ and write it as a sum of pure states in that basis with weights $p_1, p_2$...

(iv)
The set $\{\mathbb{I}, \sigma\}$ forms a basis for hermitian matrices, so we can write any $\rho$ in the form $a_0\mathbb{I} + \vec{a} \cdot \vec{\sigma}$ where all $a_i$’s are real

$$= \begin{pmatrix} a_0 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix}$$

with the constraint that $\text{Tr}(\rho) = 2a_0 = 1, a_0 = \frac{1}{2}$ and $\text{Tr}(\rho^2) \leq 1 \implies 2(a_0^2 + a_1^2 + a_2^2 + a_3^2) \leq 1$ or $|\vec{a}|^2 = a_1^2 + a_2^2 + a_3^2 \leq \frac{1}{4}$. Using $\vec{r} = \frac{\vec{a}}{2}$ gives us the expression in the question and the requirement that $|\vec{r}|^2 \leq 1$.

The state is pure iff $\text{Tr}(\rho^2) = |\vec{r}|^2 = 1$.

As the problem mentions, these vectors are called Bloch vectors and can be visualized as lying inside/on a unit sphere called the Bloch sphere with their origin at the origin of the sphere. The operators $\sim$ vectors corresponding to the pure states lie on the surface of the sphere (vectors with length = 1) while the mixed states lie everywhere else inside the sphere. This is another way to see that pure states are the extremal points, no line segment can be drawn for points in/on the sphere that has a pure state lying between the extremities of the segment.