Solutions to Problem Set 6

Physics 342

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Question 1

Spherical harmonics are the simultaneous eigenstates of the L^2 and L_z operators in the spatial angular basis. Let's label them by $|l, m\rangle$. Let $\vec{n}_i$ be a unit vector defined by $(\sin\theta_i \cos\phi_i, \sin\theta_i \sin\phi_i, \cos\theta_i)$. We can write:

$$\begin{aligned}
\mathcal{J} &= \sum_m Y^*_{lm}(\vec{n}_1) Y_{lm}(\vec{n}_2) \\
&= \sum_m \langle l, m|\vec{n}_1\rangle\langle\vec{n}_2|l, m\rangle \\
&= \sum_m \langle\vec{n}_2|l, m\rangle\langle l, m|\vec{n}_1\rangle
\end{aligned}$$

When a unitary rotation operator $U(R)$ acts on the states $|\vec{n}_i\rangle$, we get

$$\begin{aligned}
\mathcal{J}' &= \langle\vec{n}'_2|\vec{n}'_1\rangle \\
&= \sum_m \langle\vec{n}_2|U^\dagger(R)|l, m\rangle\langle l, m|U(R)|\vec{n}_1\rangle
\end{aligned}$$

Let's introduce two identity operators

$$\mathcal{J}' = \sum_m \langle\vec{n}_2| \sum_{l_2,m_2} |l_2, m_2\rangle\langle l_2, m_2|U^\dagger(R)|l, m\rangle\langle l, m|U(R) \sum_{l_1,m_1} |l_1, m_1\rangle\langle l_1, m_1|\vec{n}_1\rangle$$

Since a rotation cannot change the value of l (the magnitude of the angular momentum has to remain the same), we can write

$$\begin{aligned}
\mathcal{J}' &= \sum_m \sum_{l_2,m_2} \sum_{l_1,m_1} \langle\vec{n}_2|l_2, m_2\rangle\langle l_2, m_2|U^\dagger(R)|l, m\rangle\delta_{ll_2}\delta_{ll_1}\langle l, m|U(R)|l_1, m_1\rangle\langle l_1, m_1|\vec{n}_1\rangle \\
&= \sum_m \sum_{m_2} \sum_{m_1} \langle\vec{n}_2|l, m_2\rangle\langle l, m_2|U^\dagger(R)|l, m\rangle\langle l, m|U(R)|l, m_1\rangle\langle l, m_1|\vec{n}_1\rangle \\
&= \sum_m \sum_{m_2} \sum_{m_1} \langle\vec{n}_2|l, m_2\rangle U^{l\dagger}_{m_2 m}(R) U^l_{m m_1}(R)\langle l, m_1|\vec{n}_1\rangle
\end{aligned}$$

From unitarity of $U(R)$, we know that $\sum_m U^\dagger_{m_1 m} U_{m,m_2} = \delta_{m_1 m_2}$. Using thus in the expression above, we get

$$\begin{aligned}
\mathcal{J}' &= \sum_{m_2} \sum_{m_1} \langle \vec{n}_2 | l, m_2 \rangle \delta_{m_1 m_2} \langle l, m_1 | \vec{n}_1 \rangle \\
&= \sum_{m_2} \langle \vec{n}_2 | l, m_2 \rangle \langle l, m_2 | \vec{n}_1 \rangle \\
&= \sum_{m} \langle \vec{n}_2 | l, m \rangle \langle l, m | \vec{n}_1 \rangle \\
&= \mathcal{J}
\end{aligned}$$

Now that we know that the quantity is rotationally invariant, we can rotate the coordinate system such that one of the $\vec{n}_i$'s is along the z direction. Here I choose $\vec{n}_2$. Choosing θ to be the angle between $\vec{n}_1$ and $\vec{n}_2$, we get

$$\begin{aligned}
\mathcal{J} &= \sum_m Y^*_{lm}(\theta, \phi) Y_{lm}(0,0) \\
&= \sum_m Y^*_{lm}(\theta, \phi) \delta_{m0} \sqrt{\frac{2l+1}{4\pi}} \\
&= \sqrt{\frac{2l+1}{4\pi}} Y^*_{l0}(\theta, \phi) \\
&= \frac{2l+1}{4\pi} P_l \cos\theta
\end{aligned}$$

where I used the definition of Y_{lm} twice:

$$Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} e^{im\phi} P_l^m(\cos\theta)$$

Thus,

$$P_l \cos\theta = \frac{4\pi}{2l+1} \mathcal{J}$$

Question 2

Let's first write the wavefunction $\psi(\vec{r})$ in spherical coordinates:

$$\begin{aligned}
\psi(\vec{r}) &= (x + y + 3z) f(r) \\
&= (r \sin\theta \cos\phi + r \sin\theta \sin\phi + 3r \cos\theta) f(r) \\
&= \Omega(\theta, \phi) r f(r)
\end{aligned}$$

We can use spherical harmonics to write the angular part of the wave function and make our lives easier:

$$\begin{aligned}
\omega(\theta, \phi) &= \sqrt{\frac{2\pi}{3}} \{(Y_{1,-1} - Y_{1,1}) + i(Y_{1,-1} + Y_{1,1})\} + 6\sqrt{\frac{\pi}{3}} Y_{0,1} \\
&= (i-1)\sqrt{\frac{2\pi}{3}} Y_{1,1} + 6\sqrt{\frac{\pi}{3}} Y_{1,0} + (1+i)\sqrt{\frac{2\pi}{3}} Y_{1,-1}
\end{aligned}$$

All the spherical harmonic terms have l = 1, telling us that

$$\begin{aligned} L^2\psi(\vec{r}) &= l(l+1)\hbar^2\psi(\vec{r}) \\ &= 2\hbar^2\psi(\vec{r}) \end{aligned}$$

In other words, the state $\psi(\vec{r})$ is an eigenstate of the L^2 operator. Another way to see this would be to operate the L^2 operator directly on the wavefunction to get the same answer. Thus we will always measure $L^2 = 2\hbar^2$ with probability 1.

As the wavefunction has m = -1, 0, +1 terms, an L_z measurement can give us $-\hbar, 0 \text{and} \hbar$. For calculating probabilities, let's first calculate the absolute squared values of the coefficients:

$$\begin{aligned} |\langle m=1|\Theta\rangle|^2 &= \left|(i-1)\sqrt{\frac{2\pi}{3}}\right|^2 \\ &= \frac{4\pi}{3} \\ |\langle m=0|\Theta\rangle|^2 &= \left|6\sqrt{\frac{\pi}{3}}\right|^2 \\ &= \frac{36\pi}{3} \\ |\langle m=-1|\Theta\rangle|^2 &= \left|(i+1)\sqrt{\frac{2\pi}{3}}\right|^2 \\ &= \frac{4\pi}{3} \end{aligned}$$

The sum of these values is

$$\begin{aligned} \text{Normalization factor N} &= \frac{4\pi}{3} + \frac{36\pi}{3} + \frac{4\pi}{3} \\ &= \frac{44\pi}{3} \end{aligned}$$

Thus the probabilities are:

$$\begin{aligned} P(L_z = +\hbar) &= \frac{|\langle m=1|\Theta\rangle|^2}{N} \\ &= \frac{4}{44} \approx 0.09 \\ P(L_z = 0) &= \frac{|\langle m=0|\Theta\rangle|^2}{N} \\ &= \frac{36}{44} \approx 0.82 \\ P(L_z = -\hbar) &= \frac{|\langle m=-1|\Theta\rangle|^2}{N} \\ &= \frac{4}{44} \approx 0.09 \end{aligned}$$

Question 3

i)

The ladder operators for angular momentum generators operate as (Sakurai Eq 3.5.41),

$$\langle j', m'|J_{\pm}|j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\hbar\delta_{j'j}\delta_{m',m\pm 1}$$

Thus we get

$$\langle j = 2, m = -1|J_-|j = 2, m = 0\rangle = \sqrt{(2 + 0)(2 - 0 + 1)}\hbar = \sqrt{6}\hbar$$
$$\langle j = 1, m = 1|J_+|j = 1, m = 0\rangle = \sqrt{(1 - 0)(1 + 0 + 1)}\hbar = \sqrt{2}\hbar$$
$$\langle j = 1, m = -1|J_-|j = 2, m = 0\rangle = 0$$

ii)

To show that an angular momentum representation with $2j + 1$ states forms an irreducible representation, it would suffice to find a symmetry generator whose action outputs a state outside the subspace. Let's work in the $|m_j\rangle$ basis and assume there exists a proper subspace W that is closed under a symmetry generator action. Let k be the highest index such that $|m_k\rangle$ is not in W. Now, let's consider an arbitrary state $|\psi\rangle = \sum_i a_i|m_i\rangle \in W$. Let p be the minimum index such that $a_p \neq 0$. For the symmetry generator J_+, since $\langle m_k|J_+^{k-p}|\psi\rangle \neq 0$ and $|m_k\rangle \notin W$, we can say that the representation is indeed an irreducible representation.

iii)

From the previous part, we know that the vector spaces corresponding to j = 2 and j = 1/2 are irreducible representations. So in this 7 dimensional space, we can always choose a sub-space corresponding to the j = 2 or the j = 1/2 representation which is closed under symmetry generator actions by definition. Thus, it is a reducible representation.

Question 4

There are at least two ways of doing this. One is to use the relation for addition of angular momenta:

$$J_1 \otimes J_2 = (J_1 + J_2) \oplus (J_1 + J_2 - 1) \oplus (J_1 + J_2 - 2) ... \oplus |(J_1 - J_2)|$$

If you're not familiar with this, the section on 'The "Composite" Representation' on Pg 8 of these notes might be helpful) For the system of three spin-1/2 particles in the question, we have

$$\begin{aligned}
\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} &= (\frac{1}{2} \otimes \frac{1}{2}) \otimes \frac{1}{2} \\
&= (1 \oplus 0) \otimes \frac{1}{2} \\
&= (1 \otimes \frac{1}{2}) \oplus (0 \otimes \frac{1}{2}) \\
&= (\frac{3}{2} \oplus \frac{1}{2}) \oplus (\frac{1}{2}) \\
&= \frac{3}{2} \oplus \frac{1}{2} \oplus \frac{1}{2}
\end{aligned}$$

So the corresponding eigenstates labeled by $|S, M\rangle$ are $|\frac{3}{2}, \frac{3}{2}\rangle, |\frac{3}{2}, \frac{1}{2}\rangle, |\frac{3}{2}, -\frac{1}{2}\rangle, |\frac{3}{2}, -\frac{3}{2}\rangle, |\frac{1}{2}, \frac{1}{2}\rangle_1$, $|\frac{1}{2}, -\frac{1}{2}\rangle_1, |\frac{1}{2}, \frac{1}{2}\rangle_2, |\frac{1}{2}, -\frac{1}{2}\rangle_2$. The eigenvalues for S^2 will be $S(S+1)\hbar^2$ and S_z will be $M\hbar$. Therefore, for S^2, we get $\frac{15}{4}\hbar^2$ with multiplicity 4 and $\frac{3}{4}\hbar^2$ with multiplicity 4.

Another way is to write $\vec{S}^2$ as

$$\vec{S}^2 = \vec{S}_1^2 + \vec{S}_2^2 + \vec{S}_3^2 + 2\vec{S}_1\vec{S}_2 + 2\vec{S}_2\vec{S}_3 + 2\vec{S}_1\vec{S}_3$$

but we should be careful that these operators are acting on their respective vector spaces. It is useful to notice that

$$\begin{aligned}
2\vec{S}_i\vec{S}_j &= S_{ix}S_{jx} + S_{iy}S_{jy} + S_{iz}S_{jz} \\
&= 2S_{iz}S_{jz} + S_{i+}S_{j-} + S_{i-}S_{j+}
\end{aligned}$$

Now working in the $|S_{1z}\rangle|S_{2z}\rangle|S_{3z}\rangle$ basis, we can construct an 8×8 matrix and diagonalize it to get the same answer.