Homework 2 Solutions

Problem 1

Let $|v\rangle$ and $|w\rangle$ be any two nonzero vectors in an inner product space. We start by defining a vector orthogonal to $|w\rangle$, which will serve to project $|v\rangle$ onto a plane orthogonal to $|w\rangle$:

$$|u\rangle \equiv \left(\mathbb{I} - \frac{|w\rangle\langle w|}{\langle w|w\rangle}\right)|v\rangle \ . \tag{1}$$

One can check that indeed $\langle w|u\rangle = 0$. Now rearrange this as:

$$|v\rangle = |u\rangle + \frac{\langle w|v\rangle}{\langle w|w\rangle}|w\rangle \ , \tag{2}$$

and take the inner product with $\langle v|$ on the left side, and the expression for $|v\rangle$ in turns of $|u\rangle$ and $|w\rangle$ on the right side. We find:

$$\begin{aligned}\langle v|v\rangle &= \left(\langle u| + \frac{\langle v|w\rangle}{\langle w|w\rangle}\langle w|\right)\left(|u\rangle + \frac{\langle w|v\rangle}{\langle w|w\rangle}|w\rangle\right) \\ &= \langle u|u\rangle + \frac{\langle v|w\rangle}{\langle w|w\rangle}\langle w|u\rangle + \frac{\langle w|v\rangle}{\langle w|w\rangle}\langle u|w\rangle + \frac{|\langle w|v\rangle|^2}{\langle w|w\rangle} \\ &= \langle u|u\rangle + \frac{|\langle w|v\rangle|^2}{\langle w|w\rangle} \, ,\end{aligned} \tag{3}$$

where in the last line we have made use of the orthogonality of $|u\rangle$ and $|w\rangle$. Thus we have:

$$\langle v|v\rangle = \langle u|u\rangle + \frac{|\langle w|v\rangle|^2}{\langle w|w\rangle} \ . \tag{4}$$

Since the norm of a vector is non-negative, it's always true that:

$$\langle v|v\rangle = \langle u|u\rangle + \frac{|\langle w|v\rangle|^2}{\langle w|w\rangle} \geq \frac{|\langle w|v\rangle|^2}{\langle w|w\rangle} \, , \tag{5}$$

with the equal sign for the case that $\langle u|u\rangle = 0$. Multiplying each side by $\langle w|w\rangle$ then taking the square root, we arrive at the Cauchy-Schwarz inequality:

$$||v|| \cdot ||w|| \geq |\langle v|w\rangle| \, . \tag{6}$$

Problem 2

First, let's recall how the Gram-Schmidt procedure works. Starting from a given set of basis vectors $\{|e_i\rangle\}$, we construct a new orthogonal basis $\{|u_i\rangle\}$, where the k-th vector is determined by the previous $k-1$ vectors:

$$|u_k\rangle = |e_k\rangle - \sum_{i=1}^{k-1} \frac{\langle u_i|e_k\rangle}{\langle u_i|u_i\rangle} |u_i\rangle \, . \tag{7}$$

This guarantees the basis is orthogonal. (Note that this always gives $|u_1\rangle = |e_1\rangle$.) Finally, to get an orthonormal basis $\{|v_i\rangle\}$ we need to normalize each vector

$$|v_k\rangle = \frac{|u_k\rangle}{\sqrt{\langle u_k|u_k\rangle}} \, . \tag{8}$$

(i)

Carrying out the procedure outlined above on the vectors

$$|e_1\rangle = (1,1,1)\,, \quad |e_2\rangle = (1,1,0)\,, \quad |e_3\rangle = (1,0,1)\,, \tag{9}$$

we find the orthonormal basis:

$$|v_1\rangle = \sqrt{\frac{1}{3}}\,(1,1,1)\,, \quad |v_2\rangle = \sqrt{\frac{1}{6}}\,(1,1,-2)\,, \quad |v_3\rangle = \sqrt{\frac{1}{2}}\,(1,-1,0)\,. \tag{10}$$

(ii)

For normalizable functions on $[-1,1]$ with the standard inner product, we can start with a basis $\{|e_i\rangle\}$ with $\langle x|e_k\rangle = e_k(x) = x^k$. After applying Gram-Schmidt, we obtain the orthonormal basis:

$$p_0(x) = \sqrt{\frac{1}{2}}\,, \quad p_1(x) = \sqrt{\frac{3}{2}}x\,, \quad p_2(x) = \frac{3}{2}\sqrt{\frac{5}{2}}\left(x^2 - \frac{1}{3}\right)\,, \quad p_3(x) = \sqrt{\frac{7}{2}}\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) \ldots \tag{11}$$

(iii)

We define the weighted inner product:

$$\langle f|g\rangle = \int_{-\infty}^{\infty} f^*(x) g(x) e^{-x^2}\, dx \, . \tag{12}$$

In order to verify that this defines an inner product on functions from $(-\infty, \infty)$, we must check 3 properties:

$$\begin{aligned}&(1)\ \langle f|g\rangle = \langle g|f\rangle^* \\ &(2)\ \langle f|(a_1 g_1 + a_2 g_2)\rangle = a_1\,\langle f|g_1\rangle + a_2\,\langle f|g_2\rangle \\ &(3)\ \langle f|f\rangle \geq 0\,, \text{with } \langle f|f\rangle = 0 \text{ if and only if } f \equiv 0.\end{aligned} \tag{13}$$

(1) follows from the fact that $\left(f(x)^*\right)^* = f(x)$, while (2) follows from the linearity of integrals. (3) is a simple consequence of the fact that the integrand $|f(x)|^2 e^{-x^2}$ is positive semi-definite, and so the integral must be also. Also, the integral of a non-negative quantity can only vanish if the integrand does.

With this inner product, it is easy to compute

$$\left\langle x^2 \middle| x^2 \right\rangle = \int_{-\infty}^{\infty} (x^2)^2 e^{-x^2} \underset{\cdot}{\mathrm{x}} = \Gamma\left(\frac{5}{2}\right) = \frac{3\sqrt{\pi}}{4} < \infty\,, \tag{14}$$

so we see that x^2 is indeed normalizable. In the above, I made use of the fact that

$$\int_{-\infty}^{\infty} x^n e^{-x^2} \underset{\cdot}{\mathrm{x}} = \int_0^{\infty} y^{\frac{n-1}{2}} e^{-y} \underset{\cdot}{\mathrm{y}} = \Gamma(\frac{n+1}{2})\,, \tag{15}$$

where the Euler Gamma function $\Gamma(x)$ satisfies $x\Gamma(x) = \Gamma(x+1)$ and $\Gamma(1/2) = \sqrt{\pi}$. These facts will greatly simplify the integrals you'll need to perform below.

Again, starting with the same basis from part (ii), namely $e_k(x) = x^k$, we can build the following orthonormal basis:

$$p_0(x) = \sqrt{\frac{1}{\sqrt{\pi}}}\,,\quad p_1(x) = \sqrt{\frac{2}{\sqrt{\pi}}}\,x\,,\quad p_2(x) = \sqrt{\frac{2}{\sqrt{\pi}}}\left(x^2 - \frac{1}{2}\right) \ldots \tag{16}$$

Problem 3

We would like to prove that a matrix can be diagonalized by similarity transform if and only if it is normal. Let M be a normal matrix–*i.e.* a matrix which commutes with its Hermitian conjugate:

$$[M, M^\dagger] = 0\,. \tag{17}$$

Any matrix can be decomposed as $M = A + iB$, with A and B Hermitian. The normality condition then implies:

$$[M, M^\dagger] = [A + iB, A - iB] = -2i[A, B] = 0 \quad \Rightarrow \quad [A, B] = 0\,. \tag{18}$$

Since A and B are commuting Hermitian matrices, they can be simultaneously diagonalized by a similarity transformation:

$$D_A = U^{-1}AU\,, \quad D_B = U^{-1}BU\,, \tag{19}$$

with D_A and D_B diagonal. The fact that M is diagonalizable directly follows:

$$U^{-1}MU = U^{-1}(A + iB)U = U^{-1}AU + iU^{-1}BU = D_A + iD_B \equiv D_M\,, \tag{20}$$

with D_M diagonal.

Just for fun, let's prove the other side of the iff too. Suppose we have a matrix M which can be diagonalized by a unitary matrix U via a similarity transformation as:

$$D_M = U^{-1}MU\,, \tag{21}$$

with D_M diagonal. Taking the Hermitian conjugate of both sides, we have:

$$D_M^\dagger = \left(U^{-1}MU\right)^\dagger = U^\dagger M^\dagger (U^{-1})^\dagger = U^{-1}M^\dagger U\,, \tag{22}$$

where we've used the property $U^\dagger = U^{-1}$. We see that the same U which diagonalizes M also diagonalizes $M^\dagger$. To show M is normal, write:

$$\begin{aligned}
[M, M^\dagger] &= [UD_MU^{-1}, UD_M^\dagger U^{-1}] \\
&= UD_MU^{-1}UD_M^\dagger U^{-1} - UD_M^\dagger U^{-1}UD_MU^{-1} \\
&= UD_MD_M^\dagger U^{-1} = UD_M^\dagger D_MU^{-1} \\
&= U[D_M, D_M^\dagger]U^{-1} \\
&= 0\,,
\end{aligned} \tag{23}$$

where in the last line we've made use the fact that diagonal matrices always commute. Thus, a matrix M which can be diagonalized by a similarity transform satisfies $[M, M^\dagger] = 0$ and so is normal.

Problem 4

The inner product space $V = \{(x_1, x_2, x_3, \ldots)|\ finitely\ many\ x_i \neq 0\}$ with $\langle x|y\rangle = \sum_k x_k^* y_k$ is not a Hilbert space since it is not *complete*. To demonstrate this, we must show there exists a Cauchy sequence in V which converges to a point not in V. There are infinitely many; take for instance the following sequence:

$$\left|x^{(k)}\right\rangle = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \ldots)\,, \quad \text{where} \quad x_j^{(k)} = \begin{cases} 2^{-j/2}, & j \leq k \\ 0, & j > k \end{cases}\,. \tag{24}$$

We'll now show that $\{\left|x^{(k)}\right\rangle\}$ converges to

$$|y\rangle = (y_1, y_2, y_3, \ldots)\,, \quad \text{where} \quad y_j = 2^{-j/2}, \forall\, j \in \mathbb{N}\,. \tag{25}$$

To see this, consider

$$\lim_{k\to\infty} \left\langle y - x^{(k)} \middle| y - x^{(k)} \right\rangle = \lim_{k\to\infty} \sum_{n=k+1}^{\infty} 2^{-n} = \lim_{k\to\infty} 2^{-k} = 0\,. \tag{26}$$

So, indeed the sequence $\{\left|x^{(k)}\right\rangle\}$ converges to $|y\rangle$, but $|y\rangle \notin V$ since it contains an infinite number of non-zero entries. Thus, V is not a complete metric space, and so it is not a Hilbert space.