Solutions to Problem Set 1

Physics 342

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Note: In these solutions we use the convention that the $|0\rangle, |1\rangle$ states are eigenstates of σ_z with $|0\rangle \equiv |+\rangle$ and $|1\rangle \equiv |-\rangle$.

1 Bell states

(i) To show whether a state is entangled, we need to see whether we can write it as a product of two one-qubit states, $|\beta_{00}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$. Let's write down general $|\psi_1\rangle$ and $|\psi_2\rangle$, and then equate to $|\beta_{00}\rangle$ to constrain the general one-qubit states.

$$\begin{aligned} |\beta_{00}\rangle &= (a\,|0\rangle + b\,|1\rangle) \otimes (c\,|0\rangle + d\,|1\rangle) && (1.1)\\ &= ac\,|00\rangle + ad\,|01\rangle + bc\,|10\rangle + bd\,|11\rangle && (1.2)\\ &\equiv \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) && (1.3)\end{aligned}$$

So we have the following constraints:

$$\begin{aligned} ac &= \frac{1}{\sqrt{2}} & ad &= 0 && (1.4)\\ bc &= 0 & bd &= \frac{1}{\sqrt{2}} && (1.5)\end{aligned}$$

From the ad constraint, we know that at least one of a or d equals 0. But a cannot be 0 since ac is nonzero, and d cannot be 0 since bd is nonzero. Thus we see that there is no solution, and $|\beta_{00}\rangle$ cannot be written as a product of two single qubit states.

(ii) We can use the same approach for the other $|\beta_{xy}\rangle$.

For $|\beta_{01}\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$,

$$ac\,|00\rangle + ad\,|01\rangle + bc\,|10\rangle + bd\,|11\rangle \equiv \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{1.6}$$

gives the constraints

$$\begin{aligned} ac &= 0 & ad &= \frac{1}{\sqrt{2}} && (1.7)\\ bc &= \frac{1}{\sqrt{2}} & bd &= 0 && (1.8)\end{aligned}$$

which is impossible by the same reasoning as in part (i). You can apply this to the remaining $|\beta_{xy}\rangle$ to find that they are all entangled.

(iii) There are two ways to interpret this question. One is to take a state that is a tensor product of two general one-qubit states:

$$|\psi\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) \tag{1.9}$$

The other is to take a general two-qubit state

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \tag{1.10}$$

As we've just seen in parts (i) and (ii), these two formulations are not the same. Recall that not all two-qubit states of the second form can be written as a product of state 1 times state 2. To see this, you can just take the Bell state $|\beta_{00}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ from part (i) as an example. Thus, the first interpretation considers a subset of the states considered in the second interpretation.

Interpretation 1

The circuit we're considering first applies a Hadamard gate to bit 1, then a CNOT gate. So first, let's apply the Hadamard gate to bit 1:

$$|\psi\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) \to \frac{1}{\sqrt{2}}((a+b)|0\rangle + (a-b)|1\rangle) \otimes (c|0\rangle + d|1\rangle) \tag{1.11}$$

Now to apply the CNOT gate, let's expand out into the $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ basis:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(c(a+b)|00\rangle + d(a+b)|01\rangle + c(a-b)|10\rangle + d(a-b)|11\rangle\Big) \tag{1.12}$$

$$\to \frac{1}{\sqrt{2}}\Big(c(a+b)|00\rangle + d(a+b)|01\rangle + c(a-b)|11\rangle + d(a-b)|10\rangle\Big) \tag{1.13}$$

Let's check this with an example we have already tried. Take $|\psi\rangle = |00\rangle$, which corresponds to $a = c = 1$, $b = d = 0$. We get:

$$|00\rangle \to \frac{1}{\sqrt{2}}\Big(1|00\rangle + 0|01\rangle + 1|11\rangle + 0|10\rangle\Big) = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{1.14}$$

Interpretation 2

We'll have to deal with a few more terms for the Hadamard gate than in the first case, but otherwise the approach is very similar. The Hadamard gate will do the following:

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \tag{1.15}$$

$$\to \frac{a}{\sqrt{2}}(|00\rangle + |10\rangle) + \frac{b}{\sqrt{2}}(|01\rangle + |11\rangle) + \frac{c}{\sqrt{2}}(|00\rangle - |10\rangle) + \frac{d}{\sqrt{2}}(|01\rangle - |11\rangle) \tag{1.16}$$

$$= \frac{1}{\sqrt{2}}\Big((a+c)|00\rangle + (b+d)|01\rangle + (a-c)|10\rangle + (b-d)|11\rangle\Big) \tag{1.17}$$

Now we apply the CNOT gate:

$$|\psi\rangle \rightarrow \frac{1}{\sqrt{2}}\Big((a+c)\,|00\rangle + (b+d)\,|01\rangle + (a-c)\,|11\rangle + (b-d)\,|10\rangle\Big) \tag{1.18}$$

Again, we can check with the $|00\rangle$ case, which in this formulation is $a=1$, $b=c=d=0$:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(1\,|00\rangle + 0\,|01\rangle + 1\,|11\rangle + 0\,|10\rangle\Big) \tag{1.19}$$

$$= \frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{1.20}$$

2 Quantum circuits

Recall that each line in the diagram corresponds to a qubit, and when we write down a circuit in matrix form, we are writing down how that circuit acts on the two-qubit basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. Note that the basis states do not necessarily have to be in that order, but make sure you are clear and careful about the order in which you write down your matrix, as it will look slightly different depending on the ordering. Here I will stick with the above ordering, since that's what we've been using.

A Hadamard gate acts on a single qubit, and is represented in the $|0\rangle, |1\rangle$ basis by:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{2.1}$$

This will take $|0\rangle, |1\rangle$ and give:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \big(|0\rangle + |1\rangle \big) \tag{2.2}$$

$$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \big(|0\rangle - |1\rangle \big) \tag{2.3}$$

Circuit 1: First, look at the case where it acts on qubit 1. If we want to be thorough in our notation, we're applying the operator $H \otimes \mathbb{1}$ on the two-qubit basis states $|x_1\rangle \otimes |x_2\rangle$. Recall that the columns of a matrix correspond to the new vectors you get from the basis states (column 1 is the vector you get as a result of acting the operator on basis state 1, etc.), so we just need to see how this transformation affects the basis of the two-qubit states.

$$|00\rangle = |0\rangle \otimes |0\rangle \to \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}} \big(|00\rangle + |10\rangle \big) \tag{2.4}$$

$$|01\rangle \to \frac{1}{\sqrt{2}} \big(|01\rangle + |11\rangle \big) \tag{2.5}$$

$$|10\rangle \to \frac{1}{\sqrt{2}} \big(|00\rangle - |10\rangle \big) \tag{2.6}$$

$$|11\rangle \to \frac{1}{\sqrt{2}} \big(|01\rangle - |11\rangle \big) \tag{2.7}$$

So the matrix representing this circuit is:

$$\boxed{M_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}} \tag{2.8}$$

This looks somewhat awkward, and the action is a bit clearer when written in $|00\rangle, |10\rangle, |01\rangle, |11\rangle$ ordering since the Hadamard gate is acting on qubit 1 (leaving qubit 2 fixed). In that case it looks like:

$$M_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \tag{2.9}$$

Written this way, we can see that this is block diagonal and that there are two independent subspaces that this operator acts on: $|00\rangle, |10\rangle$ and $|01\rangle, |11\rangle$. You might have intuitively expected this based on the action of the Hadamard gate on a single qubit. Since this is block diagonal, it is relatively straightforward to check[1] that $M^\dagger M = 1$, so this is unitary.

Circuit 2: For the case where H acts on qubit 2, we can do the same analysis, now applying $\mathbb{1} \otimes H$:

$$|00\rangle \to \frac{1}{\sqrt{2}}\big(|00\rangle + |01\rangle\big) \tag{2.10}$$

$$|01\rangle \to \frac{1}{\sqrt{2}}\big(|00\rangle - |01\rangle\big) \tag{2.11}$$

$$|10\rangle \to \frac{1}{\sqrt{2}}\big(|10\rangle + |11\rangle\big) \tag{2.12}$$

$$|11\rangle \to \frac{1}{\sqrt{2}}\big(|10\rangle - |11\rangle\big) \tag{2.13}$$

So the matrix is:

$$\boxed{M_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}} \tag{2.14}$$

which is already nice and clear. There are two independent subspaces: $|00\rangle, |01\rangle$ and $|10\rangle, |11\rangle$. Again, you can check that this is unitary.

[1]Each of the 2x2 sub-matrices is unitary, and is in fact just the Hadamard matrix, which is unitary

Alternative approach: A shorter approach is to do a tensor product, if you are comfortable with them, of the identity operator and the Hadamard matrix. For the case where H is applied to qubit 1, we have:

$$\begin{aligned}
M_1 &= H \otimes \mathbb{1} && (2.15)\\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} && (2.16)\\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} && (2.17)
\end{aligned}$$

while for the case where we apply H to qubit 2,

$$\begin{aligned}
M_2 &= \mathbb{1} \otimes H && (2.18)\\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} && (2.19)\\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} && (2.20)
\end{aligned}$$

3 Positive operators

(i) We want to show that any positive operator is Hermitian. To do so, take the complex conjugate of the expectation value, noting that since the expectation value is just a complex number, this is equal to the Hermitian conjugate. Since the expectation value is real, it is equal to its complex conjugate.

$$(\langle\psi|M|\psi\rangle)^* = (\langle\psi|M|\psi\rangle)^\dagger = \langle\psi|M^\dagger|\psi\rangle \tag{3.1}$$

$$\Rightarrow \langle\psi|M^\dagger|\psi\rangle = \langle\psi|M|\psi\rangle \tag{3.2}$$

$$\langle\psi|M - M^\dagger|\psi\rangle = 0 \tag{3.3}$$

The only way $\langle\psi|M - M^\dagger|\psi\rangle = 0$ for every $|\psi\rangle$ is if $M - M^\dagger = 0$, so $M = M^\dagger$ and is Hermitian.

It must be the case that the eigenvalues of M are real and non-negative. Since M is Hermitian, we already know they are real. To see the non-negative part, note that $\langle\psi|M|\psi\rangle$ is non-negative for *any* state $|\psi\rangle$, including the eigenstates of M. If any of the eigenvalues were negative, then we could choose $|\psi\rangle$ to be the eigenstate corresponding to that eigenvalue and end up with a negative expectation value.

(ii) We want to show $M^\dagger M$ is positive for any operator M. Notice that the expectation value can be seen as the inner product of two states, $\langle\psi| M^\dagger$ and $M|\psi\rangle$. Define $|\psi'\rangle \equiv M|\psi\rangle$. Observe that

$$\langle\psi| M^\dagger = (M|\psi\rangle)^\dagger = (|\psi'\rangle)^\dagger = \langle\psi'| \tag{3.4}$$

Thus we can rewrite:

$$\langle\psi|M^\dagger M|\psi\rangle = \langle\psi'|\psi'\rangle = |\,|\psi'\rangle\,|^2 \tag{3.5}$$

The magnitude of a vector is real and greater than or equal to 0, so the operator $M^\dagger M$ is positive.

4 Expectation values from density matrix

We want to show that:

$$Tr(M |\psi\rangle \langle\psi|) = \langle\psi|M|\psi\rangle$$

To do so, let's choose a basis $|\psi_i\rangle$ for the vector space V, and write the trace in that basis. Keep in mind that $|\psi\rangle = \sum_i c_i |\psi_i\rangle$ for some complex constants c_i, and $\langle\psi| = \sum_i c_i^* \langle\psi_i|$.

$$Tr(M |\psi\rangle \langle\psi|) = \sum_i \langle\psi_i| M |\psi\rangle \langle\psi|\psi_i\rangle \tag{4.1}$$

$$= \sum_i \langle\psi_i|M|\psi\rangle c_i^* \tag{4.2}$$

$$= \sum_i c_i^* \langle\psi_i|M|\psi\rangle \tag{4.3}$$

$$= \langle\psi|M|\psi\rangle \tag{4.4}$$

In the first line, we used the definition of the trace in the $|\psi_i\rangle$ basis, and in the second and fourth lines we used the expression for $\langle\psi|$ in this basis.

Note: For a pure state, the density matrix is given by $\rho = |\psi\rangle \langle\psi|$. Thus, what we have shown in this problem (at least for pure states) is that you can find the expectation value of an operator M by $Tr(M\rho)$. This actually also generalizes to mixed states, where you have $\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$, where p_i is the probability of being in state $|\psi_i\rangle$.