Problem Set 2

Physics 342

Due January 27

Some abbreviations: S - Shankar.

1. We used the following result in our class discussion with a promise that you'd derive it in the homework. Let V be a Hilbert space with subspace W. Suppose $U : W \to V$ is a linear operator that preserves inner products. Show that there exists a unitary operator $U' : V \to V$ with the property that $U'|w\rangle = U|w\rangle$ for $|w\rangle \in W$ but with U' defined on all of V.

2. Let's follow up on the example from lecture as a way to get more comfortable with POVMs. We considered a POVM with 3 elements:

$$E_1 = \frac{\sqrt{2}}{1+\sqrt{2}}|1\rangle\langle 1|, \quad E_2 = \frac{1}{2}\left(\frac{\sqrt{2}}{1+\sqrt{2}}\right)(|0\rangle - |1\rangle)(\langle 0| - \langle 1|), \quad E_3 = \mathbf{1} - E_1 - E_2.$$

E_1 is clearly positive, and we checked that E_2 was also positive in lecture.

(i) Please check that E_3 is also positive.

(ii) Bob uses this POVM to try to determine the state $|\psi\rangle$ that Alice sends, which is either $|\psi_1\rangle = |0\rangle$ or the non-orthogonal state $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. One limitation of the POVM formalism is that it doesn't tell you the end state without knowing the measurement operators. It just tells you about the probabilities of different outcomes. Let's take measurement operators $M_i = \sqrt{E_i}$, which we mentioned in lecture. If Bob measures outcome E_3, what is the resulting state of the qubit? What happens if Bob repeats the experiment?

3. Let's generalize the preceding setup a little. Suppose Bob is given a state from m linearly independent states $|\psi_1\rangle, \ldots, |\psi_m\rangle$. Can you construct a POVM $\{E_1, \ldots, E_{m+1}\}$ so that if outcome E_i occurs with $1 \le i \le m$ then Bob knows for sure that the state is $|\psi_i\rangle$?

4. Here is another neat example of information processing using quantum mechanics. Let's start with a setup similar to the one studied in quantum teleportation: namely, Alice and Bob possess the first and second qubit, respectively, of the entangled Bell state $|\beta_{00}\rangle$. Suppose Alice has two classical bits of data that she wants to send to Bob, but she can only act on and then transmit her one qubit. Can she do it? If so, how?

5. Consider a density operator ρ. Show that $\mathrm{tr}\,(\rho^2) \le 1$ with $\mathrm{tr}\,(\rho^2) = 1$ if and only if ρ is a pure state.