Problem Set 5

Physics 342

Due February 17

Some abbreviations: S - Shankar.

1. Let's introduce the epsilon symbol ϵ_{ijk} where i, j, k run over $1, 2, 3$ or x, y, z. This is completely antisymmetric in i, j, k. Set $\epsilon_{123} = 1$ so $\epsilon_{213} = -1, \epsilon_{112} = 0$ etc. This tensor plays an important role in the study of angular momentum. Prove the two basic relations for the ϵ symbol, which we will use in lecture. Namely,

$$\epsilon_{iab}\epsilon_{icd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc},$$

and the Jacobi identity:

$$\epsilon_{ade}\epsilon_{bcd} + \epsilon_{bde}\epsilon_{cad} + \epsilon_{cde}\epsilon_{abd} = 0.$$

Here repeated indices are summed over.

2. We defined the angular momentum operators $\vec{L} = \vec{r} \times \vec{p}$. Let's call them L_i with $i = 1, 2, 3$. For a rotationally invariant system, they generate a symmetry group. That means rotation 1, represented by some unitary U_1, followed by rotation 2, represented by U_2, should be some other rotation $U_2U_1 = U_3$ which is also a symmetry. The analogous statement for the Hermitian symmetry generators L_i is that the commutator of two symmetry generators should produce a sum of symmetry generators. Let's check this.

(i) Compute $[L_i, L_j]$ in terms of ϵ_{ijk}. This is the angular momentum algebra.

(ii) Now compute $[S_i, S_j]$ for the spin 1/2 generators $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$.

(iii) Let's denote a finite rotation around some axis $\hat{n}$ by $R_{\hat{n}}(\theta)$ Here $\hat{n}$ is a unit vector pointing in some direction and $\theta \in [0, 2\pi)$. Show that any such rotation, which is an element of $SO(3)$, can be obtained from the combination $R_3(\alpha)R_2(\beta)R_3(\gamma)$ for some angles (α, β, γ) where R_i denotes rotation around the i^{th} axis. What is the range of (α, β, γ)?

(iv) Now take $R^{(1/2)}_{\hat{n}}(\theta) = e^{-\frac{i}{\hbar}\theta\, \hat{n}\cdot\vec{S}}$ using the generators of (ii). Compute the matrix $R^{(1/2)}_3(\alpha)R^{(1/2)}_2(\beta)R^{(1/2)}_3(\gamma)$ explicitly for this representation.We will explore this case more in the following question.

3. In discussing projective representations, we brought up the double cover of the rotation group $SO(3)$ which is $SU(2)$. At this point, you might be asking yourself: are $SU(2)$ and $SO(3)$ really different? You might also wonder: why do we care about the difference? To answer these questions, let us consider a map $R : SU(2) \to SO(3)$. We will use the Pauli matrices, σ^i, $i = 1, 2, 3$ to define the map. Take a 3-vector x^i and construct the combination: $X = \sum_i x^i\sigma^i$. This is a traceless Hermitian matrix for any vector $\vec{x}$. Act on X by an element $U \in SU(2)$ as follows:

$$X \to \tilde{X} = UXU^\dagger. \tag{1}$$

(i) Show that the result, $\tilde{X}$, is still a traceless Hermitian matrix.

(ii) We might wonder whether there is a 3-vector $\tilde{x}^i$ such that

$$\tilde{X} = \sum_{i=1}^{3} \tilde{x}^i \sigma^i, \tag{2}$$

and how this vector $\tilde{x}^i$ is related to x^i. To understand the relation, show that both vectors have the same length. Because the vectors have the same length, they have to be related to one another by some 3×3 rotation matrix $R(U)$ that depends on the choice of U. This is the map we want to explore. It defines what is called a "homomorphism" of groups, which means that

$$R(U_1 U_2) = R(U_1) R(U_2). \tag{3}$$

(iii) Show that the kernel of the map R is $\mathbb{Z}_2$. Conclude from this that $SO(3) = SU(2)/\mathbb{Z}_2$.

(iv) Recall that a spin 1/2 system has a two state Hilbert space with orthonormal basis vectors $|+\rangle$ and $|-\rangle$. The S_i operators of problem 2 (ii) generate rotations for the spin system. To see this consider $R_z^{(1/2)}(\phi) = e^{-iS_z\phi/\hbar}$. Conjugate S_x by this operator and see what results.

(v) Take a general ket $|\psi\rangle$ which you can expand in the $|\pm\rangle$ basis. Compute the action of $R_z^{(1/2)}(\phi)$ on this state. If you rotate by $\phi = 2\pi$, what happens to the state?

4. Let us consider the Hamiltonian

$$H = \frac{\vec{p}^2}{2m} - \frac{e^2}{r}$$

which describes a Coulomb interaction.

(i) Show that the Runge-Lenz vector

$$\vec{N} = \vec{p} \times \vec{L} - \frac{me^2}{r}\vec{r}$$

is conserved classically. Verify that $\vec{N} \cdot \vec{L} = 0$.

(ii) Show that in quantum mechanics, the vector $\vec{N}$ satisfies the following commutation relations with the angular momentum generators

$$\frac{d\vec{N}}{dt} = \left[H, \vec{N}\right] = 0, \qquad \left[L^i, N^j\right] = i\hbar\epsilon^{ijk} N^k, \qquad \left[N^i, N^j\right] = i\hbar\epsilon^{ijk}(-2mH)L^k$$

where

$$\vec{N} = \frac{1}{2}\left(\vec{p} \times \vec{L} - \vec{L} \times \vec{p}\right) - \frac{me^2}{r}\vec{r}.$$