Problem Set 6

Physics 342

Due March 3

Some abbreviations: S - Shankar.

1. Show that the quantity

$$\mathcal{J} = \sum_m Y^*_{lm}(\theta_1, \phi_1) Y_{lm}(\theta_2, \phi_2)$$

is rotationally invariant. Use the above result to prove the spherical harmonic addition theorem

$$P_l(\cos\theta) = \frac{4\pi}{2l+1} \sum_m Y^*_{lm}(\theta_1, \phi_1) Y_{lm}(\theta_2, \phi_2)$$

where θ is the angle between the directions specified by (θ_1, ϕ_1) and (θ_2, ϕ_2).

2. The wave function of a particle in three dimensions is given by

$$\psi = (x + y + 3z) f(r)$$

with $r^2 = x^2 + y^2 + z^2$. What are the possible values of L^2 and L_z which could be measured, and what is the probability of measuring each value?

3. Let's try to understand what a representation of a symmetry algebra means more precisely. A symmetry algebra consists of a set of symmetry generators. For us, these are the angular momentum generators J_i.

(i) These operators J_i are closed under commutation. They satisfy the angular momentum algebra and define what is called a Lie algebra. We built representations of the angular momentum algebra labeled by the quantum numbers of J^2 and J_z. As an exercise, compute the matrix elements

$$\langle j = 2, m = -1 | J_- | j = 2, m = 0 \rangle, \quad \langle j = 1, m = 1 | J_+ | j = 1, m = 0 \rangle,$$

$$\langle j = 1, m = -1 | J_- | j = 2, m = 0 \rangle.$$

(ii) Each angular momentum j representation with $2j + 1$ states forms an *irreducible* representation of the symmetry algebra. What this means is that the $2j + 1$ states are the basis vectors of a $2j + 1$-dimensional vector space. That these states form an irreducible representation is the statement that no non-trivial subspace of that vector space is left invariant by the action of all the symmetry generators. Show that this is true.

(iii) On the other hand, consider a vector space with basis elements given by the 5 states of $j = 2$ and the 2 states of $j = 1/2$. Show that this 7-dimensional vector space forms a *reducible* representation of the angular momentum algebra.

The intuition is that the irreps (irreducible representations) form the building blocks of general representations.

4. Three spin 1/2 particles have spins $\vec{S}_1, \vec{S}_2, \vec{S}_3$. What are the possible eigenvalues of $\vec{S}^2$ where $\vec{S} = \vec{S}_1 + \vec{S}_2 + \vec{S}_3$? What are the multiplicities of each eigenvalue?