6.2 Decomposing tensor products

Suppose have 2 transform τ_1 τ_2 spin j_1 j_2 respect.

How does $\tau_1 \tau_2$ transform?

Decompose $|j_1\rangle \otimes |j_2\rangle$ into irreps.

$$\underset{V_1}{\|}\qquad\qquad \underset{V_2}{\|}$$

group acts by $(T_1 \otimes T_2)(g)\,(v_1 \otimes v_2) = (T_1(g)v_1) \otimes (T_2(g)v_2)$

$\Rightarrow$ algebra acts by $J^a(|j_1 m_1\rangle \otimes |j_2 m_2\rangle) =$
$(J^a|j_1 m_1\rangle \otimes |j_2 m_2\rangle) + (|j_1 m_1\rangle \otimes J^a|j_2 m_2\rangle)$

In particular J^3 values add.

$\Rightarrow$ unique highest weight state $j = j_1 + j_2$

step down

$$\begin{aligned} J^-|j, j\rangle &= N_j |j, j-1\rangle \\ &= J^-(|j_1 j_1\rangle \otimes |j_2 j_2\rangle) \\ &= N_{j_1}(|j_1\, j_1-1\rangle|j_2\, j_2\rangle) + N_{j_2}(|j_1 j_1\rangle|j_2\, j_2-1\rangle) \end{aligned}$$

etc. gives $2j+1$ states.

Remaining states are orthogonal and so there must be a highest weight. Continue to reduce.

Eventually find all reps.

6.3 Wigner – Eckart thm.

A tensor operator O^j_l $l = 1 \ldots 2j+1$ is a set of operators $\ni$ $[J_a, O^j_l] = O^j_{l'} (J_a)_{l'l}$

irrep j

w/ orbital ang. mom.

$$J_a = \epsilon_{abc} x_b p_c$$

$$[J_a, x_b] = \epsilon_{acd} [x_c p_d, x_b]$$

$$= -i \underbrace{\epsilon_{acb}}_{\text{structure constant}} r_c$$

$$= r_c [J_a^{adj}]_{cb}$$

Spin 1 for $\vec{x}$!

Note $O^{j_1}_l | j_2, m\rangle$ transforms as $j_1 \otimes j_2$

To see this, apply

$$J^a (O | j_2 m\rangle) = [J^a, O^{j_1}_l] | j_2 m\rangle + O^{j_1}_l J_a | j_2 m\rangle$$

Again J^3 adds $(l+m)$ eigenvalue.

we can decompose into irreps. Just the tensor product of 2 irreps.

Each rep from j_1+j_2 to $|j_1-j_2|$ appears once!
(HWK).

$\Rightarrow$ decompose $O_l^{j_1} \; |j_2, m, \alpha\rangle$ into these reps.

collection of states with potentially funny normalization. Hence α all other observables.

$$O_l^{j_1} \; |j_2 \; m \; \alpha\rangle = \sum_{J=|j_1-j_2|}^{j_1+j_2} \langle J, l+m \,|\, j_1, j_2, l, m\rangle \cdot k_J \,|J, l+m\rangle$$

J^3 values add

indep. of m

What are these coefficient?

$\langle J, l+m \,|\, j_1, j_2, l, m\rangle$ means the coefficient

of $|J, l+m\rangle$ in the product $|j_1 \, l\rangle |j_2 \, m\rangle$

Clebsch – Gordon coefficients.

Determined by group theory.

Only ambiguity is a choice of phase for these states.

Need to express $|J, l+m\rangle$ in terms of $|J, l+m, \beta\rangle$ Hilbert space basis states.

$$k_J \,|J, l+m\rangle = \sum_\beta k_{\alpha\beta} \,|J, l+m, \beta\rangle$$

↖ unknown

unknown. Depend on α, j_1, O^{j_1} and j_2 (also β, J)

Not on l or m

$k_{\alpha\beta}$ are reduced matrix elements

$$k_{\alpha\beta} = \langle J, \beta \,|\, O^{j_1} \,|\, j_2\, \alpha\rangle$$

Thm: (Wigner – Eckart)

$$\langle J, m', \beta \,|\, O^{j_1}_l \,|\, j_2, m, \alpha\rangle =$$

$$\delta_{m', l+m} \langle J, l+m \,|\, j_1, j_2, l, m\rangle \cdot \langle J, \beta \,|\, O^{j_1} \,|\, j_2, \alpha\rangle$$

If we know any non-zero matrix element of O between states of given J, β and j_2, α we get all the rest using symmetry.

eg/ Suppose $\langle \frac{1}{2}, \frac{1}{2}, \alpha \,|\, r_3 \,|\, \frac{1}{2}, \frac{1}{2}, \beta\rangle = A$

what is $\langle \frac{1}{2}\, \frac{1}{2}\, \alpha \,|\, r_1 \,|\, \frac{1}{2}\, -\frac{1}{2}\, \beta\rangle = ?$

First decompose $\vec{r}$ into definite J^3 eigenstates

$$[J^3, r_3] = 0 \qquad [J^\pm, r_3] = \mp \frac{(r_1 \pm i r_2)}{\sqrt{2}} = r_{\pm 1}$$

$$\Rightarrow \quad r_1 = \frac{1}{\sqrt{2}}(-r_{+1} + r_{-1}) \qquad \text{spin } 1$$

$\Rightarrow \langle \tfrac{1}{2}\ \tfrac{1}{2}\ \alpha \,|\, q_1 \,|\, \tfrac{1}{2}\ {-\tfrac{1}{2}}\ \beta\rangle =$

$\langle \tfrac{1}{2}\ \tfrac{1}{2}\ \alpha \,|\, \tfrac{1}{\sqrt{2}}\left(-r_{+1} + r_{-1}\right) |\, \tfrac{1}{2}, -\tfrac{1}{2}, \beta\rangle$

dies

$= \langle \tfrac{1}{2}\ \tfrac{1}{2}\ \alpha \,|\, \tfrac{-1}{\sqrt{2}}\, r_{+1} \,|\, \tfrac{1}{2}, -\tfrac{1}{2}, \beta\rangle$

Use Wigner - Eckhart but need to look up Clebsch - Gordon.

$\langle \tfrac{1}{2}, \tfrac{1}{2} \,|\, 1, \tfrac{1}{2}, 1, -\tfrac{1}{2}\rangle$

It's actually A! Power of group theory.

Next weights & roots.

6.4 Roots & Weights

Let's generalize the SU(2) construction to SU(N).

Suppose we have a Lie algebra g

$$[\vec{e}_i, \vec{e}_j] = \sum_k f^k_{ij} \vec{e}_k$$

Hermitian (physics convention)

structure consts.

Def 6.4.1 A ~~the~~ maximal set of commuting generators span the Cartan subalgebra (or maximal torus).

ie take the most commuting operators, H_i $i = 1, \ldots,$ rank $g = m$ which satisfy

$$H_i^\dagger = H_i \qquad [H_i, H_j] = 0$$

In an irrep D, we can choose to normalize the Cartan generators, H_i, $\ni$ $\mathrm{Tr}(H_i H_j) = k_D \delta_{ij}$

normalization

Simultaneously diagonalize H_i. Generalize "J_z" of SU(2)

$$J_z |j, m\rangle = m |j, m\rangle$$

Analogously $H_i |\mu, D\rangle = \mu_i |\mu, D\rangle$
$\uparrow$ rep

Def. 6.4.2 The μ_i are weight vectors for the D^{th} rep. They are (real) vectors in a rk G dimensional space.

Use vector notation $\alpha \cdot \mu \equiv \alpha_i \mu_i$ $\quad \alpha^2 \equiv \alpha_i \alpha_i$

w/ $SU(2)$ rk $= 1$ e.v. of J_z

Recall that the adj. rep is the dim of the group and acts by ad.

Label states by the generators of g

$$X_a \longrightarrow |X_a\rangle$$

$$X_b |X_a\rangle = |[X_b, X_a]\rangle$$

Choose a norm $\quad \langle X_a | X_b \rangle = \frac{1}{\lambda} \operatorname{tr} X_a^{+} X_b$

Def 6.4.3 The roots are the weights of the adj. representation.

What about the Cartan generators themselves

$$H_i | H_j \rangle = | [H_i, H_j] \rangle = 0$$

Zero weight vectors correspond to Cartan elements.

For every other element,

$$H_i \, |E_\alpha\rangle = \alpha_i \, |E_\alpha\rangle$$

$$\Rightarrow \quad [H_i, E_\alpha] = \alpha_i E_\alpha$$

E_α are like J^+, J^-. Not Hermitian

$$[H_i, E_\alpha^+] = -\alpha_i E_\alpha^+ \qquad E_\alpha^+ = E_{-\alpha}$$

States with different weights are orthogonal

So we can choose

$$\langle E_\alpha | E_\beta \rangle = \lambda^{-1} \, \mathrm{tr} \, E_\alpha^+ E_\beta = \delta_{\alpha\beta} \quad \left(= \prod_i \delta_{\alpha_i \beta_i}\right)$$

We now raise and lower as before.

any rep. $\left\{ \begin{array}{l} E_{\pm\alpha} \, |\mu, D\rangle \text{ has weight } \mu \pm \alpha \\ H_i E_{\pm\alpha} \, |\mu, D\rangle = [H_i, E_{\pm\alpha}] \, |\mu, D\rangle + E_{\pm\alpha} H_i |\mu, D\rangle \\ \qquad = (\mu \pm \alpha) \, E_{\pm\alpha} |\mu, D\rangle \end{array} \right.$

Apply to adjoint

$E_\alpha \, |E_{-\alpha}\rangle$ weight 0 $\Rightarrow$ combination of Cartan elements

$$\Rightarrow \quad [E_\alpha, E_{-\alpha}] = \cancel{c_i |H_i\rangle} \quad c_i H_i$$

What is c_i?

$$\begin{aligned} c_i &= \langle H_i | E_\alpha | E_{-\alpha} \rangle \\ &= \lambda^{-1} \, \mathrm{tr} \left(H_i [E_\alpha, E_{-\alpha}] \right) \\ &= \lambda^{-1} \, \mathrm{tr} \left(E_{-\alpha} [H_i, E_\alpha] \right) \\ &= \alpha_i \, \lambda^{-1} \, \mathrm{tr} \left(E_\alpha^+ E_\alpha \right) = \alpha_i \end{aligned}$$

$\Rightarrow \quad c_i = \alpha_i \quad$ and $\quad [E_\alpha, E_{-\alpha}] = \alpha \cdot H$

just like $\quad [J^+, J^-] = J^3$.

6.5 $su(2)$ subalgebras

$E_\alpha \neq 0$ then $E_{\pm\alpha}$ can be used to construct an $su(2)$ subalgebra.

$$E^{\pm} = |\alpha|^{-1} E_{\pm\alpha} \qquad E^3 \equiv |\alpha|^{-2} \alpha \cdot H$$

HWK check.

An irrep of $G \xrightarrow{\text{decomposes}}$ irreps of $su(2)$

Can use this to constrain weights.

$$E_3 |\mu, x, D\rangle = \frac{\alpha \cdot \mu}{\alpha^2} |\mu, x, D\rangle$$

other labels beyond weight

$su(2)$ rep. $\Rightarrow \quad \dfrac{2\alpha \cdot \mu}{\alpha^2} \in \mathbb{Z}$

A general state can always be written as a lin. combination of states transforming in irreps of this $su(2)$.

Suppose highest spin state in lin. combination is j

$\Rightarrow \quad (E^+)^p |\mu, x, D\rangle \neq 0$ for some p is the highest weight vector for E_3 (weight $\mu + p$)

$$(E^+)^{p+1} |\mu, x, D\rangle = 0$$

E_3 value is $\frac{\alpha \cdot (\mu + p\alpha)}{\alpha^2} = \frac{\alpha \cdot \mu}{\alpha^2} + p = j$

Same for lowering $(E^-)^q |\mu, x, D\rangle \neq 0$ where $q+1$ vanishes. Weight is $\mu - q\alpha$

E_3 value is,

$$\frac{\alpha \cdot (\mu - q\alpha)}{\alpha^2} = \frac{\alpha \cdot \mu}{\alpha^2} - q = -j$$

Adding gives $\frac{\alpha \cdot \mu}{\alpha^2} = -\frac{1}{2}(p - q)$ $(*)$

Gives a geometric classification of Lie groups.

Take 2 roots α, β. For E_α we get $(*)$

~~while for E_β~~ $\cancel{\beta}$

$\frac{\alpha \cdot \beta}{\alpha^2} = -\frac{1}{2}(p - q)$ while for E_β

$\frac{\beta \cdot \alpha}{\beta^2} = -\frac{1}{2}(p' - q')$ $\alpha \cdot \beta = |\alpha| |\beta| \cos \theta_{\alpha\beta}$

multiplying

$$\Rightarrow \cos^2 \theta_{\alpha\beta} = \frac{(\alpha \cdot \beta)^2}{\alpha^2 \beta^2} = \frac{\overbrace{(p-q)(p'-q')}^{\text{integer!}}}{4}$$

up to complements

$(p-q)(p'-q')$	$\theta_{\alpha\beta}$
0	90°
1	60, 120°
2	45, 135°
3	30, 150°

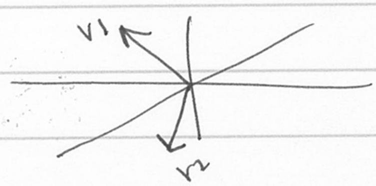
What about 4? 0° or 180°

However roots are unique. No non-zero multiple of α (except $-\alpha$) is a root.

$\Rightarrow$ $\alpha = \beta$ or $\alpha = -\beta$. Not interesting.

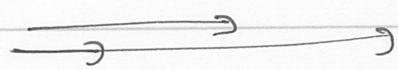
6.6 Simple Roots

"Need a notion of highest weight"



$v1 > v^2$?
$v1 < v^2$? How do tell?

$SU(2)$



clear.

Def 6.6.1: A weight is positive if its first non-zero component is positive. If negative, it is negative for a fixed basis of the Cartan sub-algebra.

Seems basis ~~indep~~ dep. but the results do not depend on the basis.

ex/ $(0, 0, 0, -\frac{1}{\sqrt{3}}, \ldots)$ is negative
$(0, 1, 0, \ldots)$ is positive

$\vec{\mu} > \vec{\nu}$ if $\mu - \nu$ is positive

Now have a notion of highest weight.
In the adjoint rep., pos. roots correspond to raising ops while negative roots lower.

$$E_\alpha \,|\text{highest weight}\rangle = 0 \quad \alpha > 0$$

To simplify life, we define

<u>Def 6.6.2</u>: A root is simple if it is positive and cannot be written as a sum of positive roots.

i) α, β simple $\quad \alpha - \beta$ is not a root.

Pf. Either $\alpha-\beta$ or $\beta-\alpha$ is pos. so $(\alpha-\beta)$
$\alpha = (\alpha-\beta) + \beta$ contradiction.

ii) As a consequence $\quad E_{-\alpha}|\beta\rangle = E_{-\alpha}|E_\beta\rangle$
$$= E_{-\beta}|E_\alpha\rangle = 0$$

From $\quad \dfrac{\alpha\cdot\mu}{\alpha^2} = -\frac{1}{2}(p-q) \qquad (*)$

$$\Rightarrow \quad \frac{\alpha\cdot\beta}{\alpha^2} = -\frac{1}{2}(p-q) \longleftarrow q=0$$
$|E_\beta\rangle$ is killed by $(E_{-\alpha})^{q+1}$ times.

<u>H.W.K.</u> Show $[E_\alpha, E_\beta] \sim E_{\alpha+\beta}$. What if $\alpha+\beta$ not a root?

Similarly $\quad \dfrac{\beta\cdot\alpha}{\beta^2} = -\frac{1}{2}(p'-q') \qquad q'=0 \quad (E_{-\beta})^{q}|E_\alpha\rangle =$

Knowing p and p' for each simple root tells us the angles between them and their relative lengths

$$\cos\theta_{\alpha\beta} = -\frac{\sqrt{\rho\rho'}}{2} \qquad \frac{\beta^2}{\alpha^2} = \rho/\rho'$$

iii) $\underbrace{\frac{\pi}{2} \leq \theta}_{\text{cosine negative}} < \pi \longleftarrow$ roots are positive

$\Rightarrow$ simple roots are linearly independent.

iv) Any positive root $\phi = \sum_\alpha k_\alpha \alpha$ k_α non-neg. integers α simple roots.

v) There are rk $G = m$ simple roots. Span the m-dimensional root space.

Pf: Suppose not true. $\exists\ \vec{v}$ orthogonal to simple roots. $[\vec{v}\cdot\vec{H}, E_\phi] = 0$ all roots ϕ

$\Rightarrow$ $\vec{v}\cdot\vec{H}$ commutes with everything so there is an extra $u(1)$ factor — not simple.

vi) Get all the roots from the simple roots.

$\phi_k = \sum k_\alpha \alpha$ $\quad k = \sum k_\alpha$ which ϕ_k are roots?

All ϕ_1's are roots (simple roots)

Know all roots for $k \leq l$ then we look at

$E_\alpha|\phi_l\rangle$ for all α to find all ϕ_{l+1}

$$\frac{2\alpha\cdot\phi_l}{\alpha^2} = -(p-q).$$

Know q by induction since we know how we built ϕ_l so we get p. If $p>0$ then $\phi_l+\alpha$ is a root.

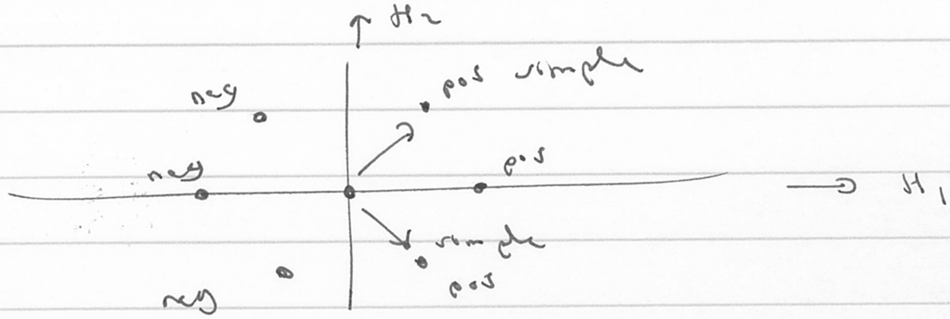
ex/ $p = 1$ start with $\phi_1 = \beta \leftarrow$ simple root.

q's $= 0$ $\qquad \dfrac{2\alpha \cdot \phi_1}{\alpha^2} = \dfrac{2\alpha\cdot\beta}{\alpha^2} = -p$

$\alpha \cdot \beta = 0 \Rightarrow p = 0$ $\quad \alpha + \beta$ not a root. otherwise

$p > 0$ $\quad \alpha + \beta =$ a root. (non-trivial raising operator)

ex/ su(3) HWK



$(1,0) = \alpha^1 + \alpha^2$ $\qquad \alpha^1 = (\tfrac{1}{2}, \sqrt{3}/2)$

$\alpha^2 = (1/2, -\sqrt{3}/2)$

$(\alpha^1)^2 = (\alpha^2)^2 = 1$ $\qquad \alpha^1 \cdot \alpha^2 = -1/2$

$$\frac{2\alpha^1 \cdot \alpha^2}{(\alpha^1)^2} = \frac{2\alpha^1\cdot\alpha^2}{(\alpha^2)^2} = -1 \quad \text{so } p = 1$$

$\Rightarrow$ $\alpha_1 + \alpha_2$ is a root.

6.7 Dynkin Diagrams

What's a smart way of encoding the data about simple roots : a Dynkin diagram!

o o	angle $90°$	
o—o	angle $120°$	nodes of a Dynkin diagrams
o=o	angle $135°$	
o≡o	$150°$	

w/ o $su(2)$ o—o $su(3)$ o—o—o $su(4)$

Are all roots the same length?

Def 6.7.1 The "Cartan matrix" has entries

$$A_{ji} = \frac{2\alpha^j \cdot \alpha^i}{(\alpha^i)^2}$$

where α^i is a simple root. It ~~is ant~~ has 2's on the diagonal, and off-diagonal entries are $0, -1, -2,$ or -3.

Each simple root generates an $su(2)$. What Cartan tells us is how one simple root transforms as a rep. w.r.t. to the $su(2)$'s of other simple roots.

The A_{ji} element is the $(q-p)$ value for α^i acting $|\alpha^j\rangle$; twice the E_3 value hence integer.

$$2E_3|\mu\rangle = \frac{2H\cdot\alpha^i}{(\alpha^i)^2}|\mu\rangle = \frac{2\mu\cdot\alpha^i}{\alpha_i^2}|\mu\rangle = (q^i-p^i)|\mu\rangle$$

$\leftarrow$ vector dot product

For pos. root $\phi = \sum k_j \alpha^j$

$$q^i - p^i = \frac{2\phi\cdot\alpha^i}{(\alpha^i)^2} = \sum_j k_j \frac{2\alpha^j\cdot\alpha^i}{(\alpha^i)^2}$$

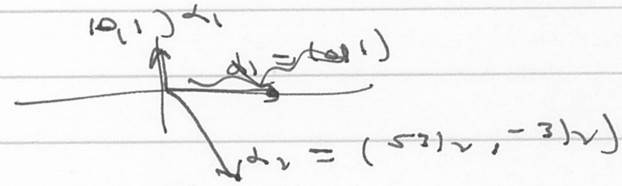
$$= \sum_j k_j A_{ji}$$

So diagonal is 2 because $E_3 = 1$ for each simple root itself (spin 1)

jth row of Cartan consists of $q_i - p_i$ values of the simple roots α^j.

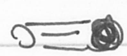
ex/ $su(3)$ $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ has relative lengths!

what is $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ (G_2 algebra)



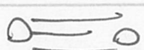
$(\alpha^1)^2 = 1$ $(\alpha^2)^2 = 3$

$\frac{2\alpha^1\cdot\alpha^2}{(\alpha^1)^2} = -3$ $\frac{2\alpha^1\cdot\alpha^2}{(\alpha^2)^2} = -$



The length & angle of simple roots characterize a Lie algebra. Simple Lie algebras at most have roots of 2 lengths. In these cases

O ← longer roots • shorter roots

 -3 ratio $\sqrt{3}$

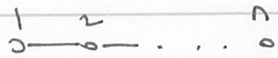
 -2 $\sqrt{2}$

 -1 1

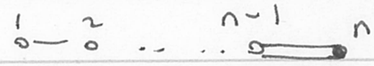
follows from Cartan or Dynkin diagram.

A list

A_n $(SU(N+1))$

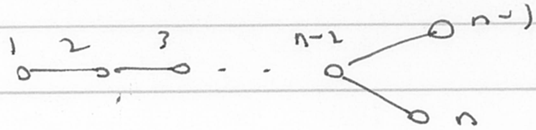


B_n $(SO(2n+1))$

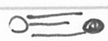


C_n $(Sp(2n))$

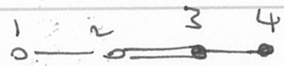
D_n $(SO(2n))$



G_2



F_4

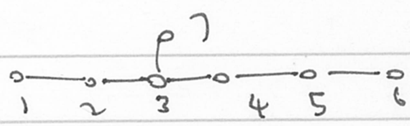


$\xrightarrow{\text{Cartan}}$

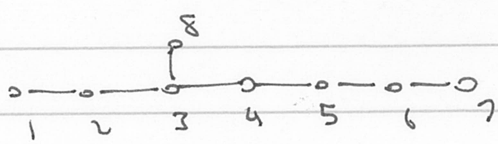
$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

E_6

E_7



E_8



6.8 Dynkin Coefficients

Take irrep D then highest weight μ satisfies $\mu + \phi$ is not a weight for any positive root ϕ.

$\Rightarrow \quad E_{\alpha^j} |\mu\rangle = 0 \quad \forall j \quad \alpha^j$ simple root
$j = 1 \dots m \leftarrow \operatorname{rk} G$

Actually "iff" statement. Defines highest weight.

For every E_{α^j} acting on $|\mu\rangle$ $\quad p = 0$

$$\frac{2\alpha^j \cdot \mu}{(\alpha^j)^2} = \ell^j \leftarrow \text{non-neg. integer}$$

Def 6.8.1 The ℓ^j label irreps $(j = 1 \dots m)$ "Dynkin coefficients"

For the special case of μ^j satisfying

$$\frac{2\alpha^j \cdot \mu^k}{(\alpha^j)^2} = \delta_{jk},$$

the μ^j are called fundamental weights.

Every highest weight can be uniquely written

$$\mu = \sum_{j=1}^{m} \ell^j \mu^j$$

A vector with 1 non-zero entry.

The m irreducible reps with these weights are fundamental representations. Sometimes denoted D^j.

Lastly $\ell^j = q^j - p^j$ for simple root α^j.

(Lost Young tableaux & tensor products. (Slansky))

You can now use precisely the same highest weight algorithms!

7. Young Tableaux & Tensor Methods

7.1 Recall that S_n has one irrep for each conjugacy class?

How many conjugacy classes?

i) If $(i_1 \dots i_k)$ is a cycle of length k then $g(i_1 \dots i_k)g^{-1}$ is of length k

ii) Any 2 cycles of length k are conjugate

iii) Any permutation can be written as a product of disjoint cycles.

$$\sigma = (12)(34)(10,11)(56789) \in S_{11}$$

Denote conjugacy class by $(1)^{\nu_1}(2)^{\nu_2}(3)^{\nu_3}\dots(n)^{\nu_n}$

ν_j number of cycles of length j.

$\Rightarrow$ Irreps of S_n are in 1-1 correspondence with the partitions of n.

ex/ $11 = 0 + 3(2) + 1(5)$ partition of 11

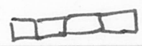
7.2 Young tableaux

Method for encoding a partition of n

$$
\begin{aligned}
4 &= 4 \\
&= 3+1 \\
&= 2+2 \\
&= 2+1+1 \\
&= 1+1+1+1 \qquad p(4) = 5
\end{aligned}
$$

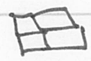
① Draw a j-cycle by a column of length j

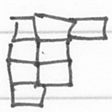
② Decreasing j as you go right.

 $\leftarrow 1+1+1+1$

e.g./  $\leftarrow 4$

 $3+1$ $\quad \in S_4$

 $2+1+1$  $2+2$

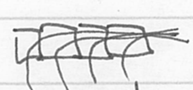
 $4+3+1 \in S_8$

1-1 correspondence with irreps of S_n

What is the dimension of the rep?

"use the hooks rule"

$$\frac{n!}{H} \leftarrow \text{hooks factor}$$



4 3 2 1 $\qquad \dfrac{4!}{4!} = 1$

4 2 1 $\qquad \dfrac{4!}{4 \cdot 2 \cdot 1} = 3$

One hook for each box. The number of boxes each hook passes through h_i $\qquad H = \prod_{i=1}^{n} h_i$

$$\frac{4!}{4\cdot 2} = 3 \quad \text{dim.}$$

How do we construct the rep?

Assign integers to each box $1 \ldots n$

$n!$ ways

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 \\ \cline{1-1}\end{array} \longrightarrow (12453) \quad \text{pick an order}$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \longrightarrow (1243)$$

Now symmetrize in rows / antisymmetrize in columns

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \rightarrow (12) + (21)$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \rightarrow (123) + (213) - (321) - (231)$$

These are our vectors. We know how S_n acts on these vectors by permuting our initial assignment of numbers.

w/ (123) $\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \xrightarrow{\text{state}} (123) + (213) - (321) - (23$

Act by $(12)(3) \in S_3$ on state (123)

$$\hookrightarrow \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \xrightarrow{\text{state}} (213) + (123) - (312) - (321)$$

(1234) $\quad \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 1 & 2 \\ \hline\end{array} \xrightarrow{\text{state}} (4312) + \ldots$

Act by $(12)(34) \in S_4$ on state (4312)

$$\hookrightarrow \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 2 & 1 \\ \hline\end{array} \xrightarrow{\text{state}} (3421) + \ldots$$

Rep. is irreducible! Way to build S_n irreps.

7.3 tensor methods (short version)

Def 7.3.1 An invariant tensor does not transform under a G-transformation.

w/ δ^i_j for $SO(N)$ $i=1\ldots N$

ϵ_{ijk} for $SO(3)$

$\epsilon_{\alpha\beta}$ for $SU(2)$ ~~and $SU(N)$~~

7.3.1 Irreps of $SU(2)$ (ref: Coleman "Aspects of Symmetry")

$i=0,1$ (tensoring many spin 1/2)

$T_{i_1\ldots i_k}{}^{j_1\ldots j_k} \xrightarrow[\text{with } \epsilon_{i_1 i_2}]{\text{lower}} T_{i_1\ldots i_n}$ ← only have to consid

T_{ij} is reducible in general $T_{ij} = T_{\{ij\}} + T_{[ij]}$

$T_{[ij]} = \epsilon_{ij}\, T$ ← forget this, lower rank

$\Rightarrow$ consider symmetric tensors $T_{i_1\ldots i_n}$

T_{i_1} $i_1=0,1$ 2-dim. spin 1/2

$T_{i_1 i_2}$ (00) (01) (11) 3-dim spin 1

$T_{i_1\ldots i_{2s}}$ spin s dimensional.

~~$v^\alpha \otimes w^\beta$~~

$v^\alpha w^\beta$ denote by dimension $v^\alpha w^\beta \epsilon_{\alpha\beta}$

$2\otimes 2 = 3 \oplus 1$ $v\, 6^{\alpha} w$

$\frac{1}{2}$ $\frac{1}{2}$

7.3.2 Same for $SU(3)$ $\epsilon_{i_1 i_2 i_3}$ invariant tensor

Need to consider up and down indices

$T^{i_1 \cdots i_l}_{j_1 \cdots j_m}$ $\Rightarrow$ can still reduce antisymm. part using epsilon

$\Rightarrow$ consider symmetric in upper / lower indices and traceless.

T^{i_1} T_{i_1} are 3 and $\overline{3}$ cmplx conj. reps.

Under $SU(3)$ $U^{i_1}{}_{i_2} T^{i_2}$ $U_{i_1}{}^{i_2} T_{i_2}$

$U_{i_1}{}^{i_2} = (U^{i_1}{}_{i_2})^*$

$T^{i_1 \cdots i_n}_{j_1 \cdots j_m}$ (n,m) rep. of $SU(3)$

$\overline{(n,m)} = (m,n)$

dim is combinatorics $\dim(n,m) = \frac{1}{2}(n+1)(m+1)(n+m+2)$

$(1,0)$ 3
$(0,1)$ $\overline{3}$
$(1,1)$ 8
$(3,0)$ 10
$(2,2)$ 27 etc.

label by dimension.

and so on for $SU(N)$.

Symmetries of indices label irreps of $SU(3)$

$S_n \longleftrightarrow SU(3)$

Raise all indices.

 ϵ^{ijk}  $T^{i_1 \cdots i_4} = 0$ in $SU(3)$ anti.

ex)  $T^{j_1 j_2 k_1}$ symm. j_1, j_2 antisymm. j_1, k_1

$(1,1)$ $T^{j_1 j_2 k_1} = \epsilon^{j_1 k_1 l}\, T_l^{\ j_2} + j_2 \leftrightarrow j_1$

7.3.3 Clebsch–Gordan Algorithm.

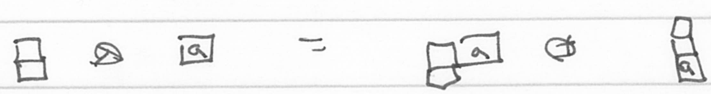
Decompose tensor product of irreps A and B using Young tableaux. Build on A using boxes from B.

Put a's in top row of B, b's in second row etc. Take boxes from top row of B and add to A building to the right and/or down to form a tableaux, with no 2 a's in same column (antisymmetry!)

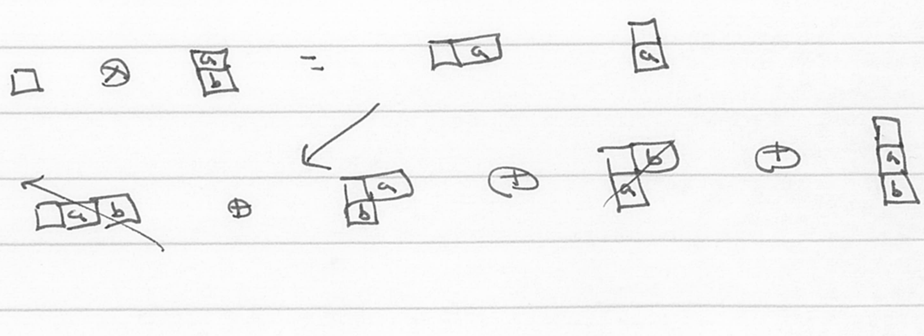
Do same with second row with one condition: Along each row from right to left from top row down to bottom row, the number of a's $\geq$ number of b's.

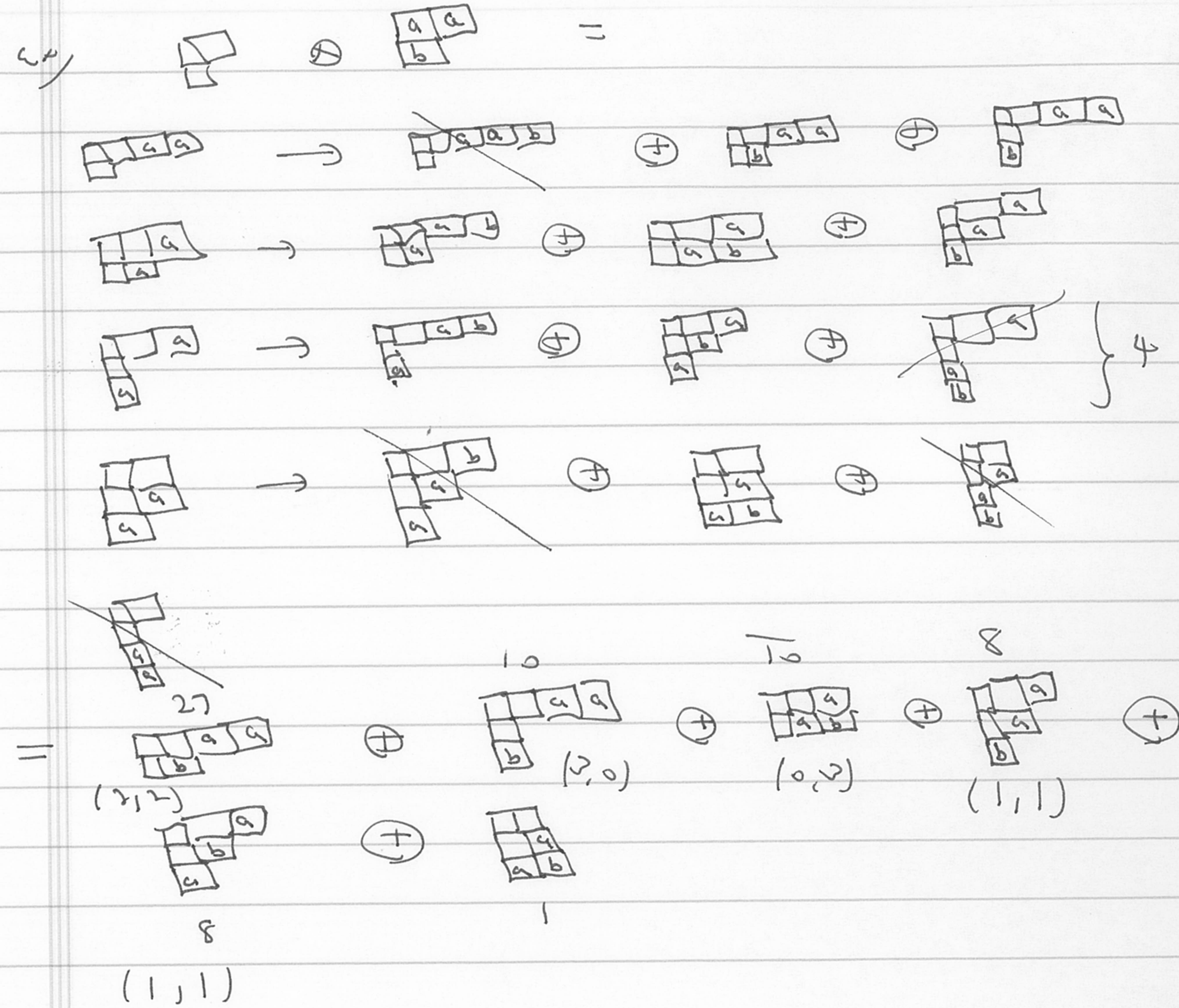
Forget 3 columns for su(3)

$3 \times 3 = 6 \oplus \bar{3}$



$\bar{3} \otimes 3 = 8 \oplus 1$





Dimensions from tensor rule!

Much more for another course . . . .

Can summarize by saying irreps of SU(N) are irreps of S_n.

(Schur – Weyl duality)

Gives algorithm for Clebsch – Gordan decomp.