

Generalized actions in odd elasticity and quantum field theory

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1 Introduction

Active matter refers to physical and biological systems in which the basic microscopic constituents are able to metabolize internal sources of energy. Examples range from cell membranes [1] and flocks of animals [2] to self-propelled colloids [3] and interacting robots [4]. The dynamics of these phenomena raise physical questions strikingly similar to those encountered in quantum field theory.

In this paper, we focus on *odd elasticity*, which describes active solids in which the elastic stiffness tensor acquires an anti-symmetric contribution that would otherwise be forbidden by energy conservation [5]. As we will show, this modification introduces terms into the equations of motion that do not follow from the variation of a typical action, i.e. the integral over space and time of $T - V$, where T is the kinetic energy density and V is the potential energy density. The central question we ask in this paper is:

Is there a generalized notion of an action S whose variation $\delta S = 0$ gives rise to the equations for motion of an odd elastic solid?

Specifically, we explore techniques from quantum field theory that originated in the context of the Wess-Zumino action [6]. The quantum mechanical systems of interest share an important similarity with odd elasticity: antisymmetric terms in the equations of motion function to break undesired symmetries, and these terms do not follow from a standard Lagrangian density. Nonetheless, for these systems, a topological term that is integrated over a higher dimensional region in field space can be added to the action to capture the desired physics.

We will examine the similarities between the structures in QFT and odd elasticity and discuss the complications that arise from straight-forward attempts to apply the QFT techniques to active fields. A variational principle for odd elasticity (and other active systems) remains an open question, and this paper highlights a direction that may be useful with further investigation.

2 Odd elasticity

Elasticity is the continuum description of the relationship between stress and strain in solid bodies. When a solid is deformed, a point originally at location x_i is mapped to a new location $X_i(\mathbf{x})$. The difference $u_i(\mathbf{x}) \equiv X_i(\mathbf{x}) - x_i$ is known as the displacement field. The solid body resists changes of shape and volume, and so the gradients in displacement $u_{ij}(\mathbf{x}) \equiv \partial_i u_j(\mathbf{x})$ induce internal forces described by a stress field $\sigma_{ij}(\mathbf{x})$. In the case of linear elasticity, valid for strains, Hooke's law is the basic assumption that the induced stresses are linearly proportional to the gradients of displacement:

$$\sigma_{ij} = K_{ijmn} u_{mn}, \quad (1)$$

where K_{ijmn} is known as the elastic (or stiffness) tensor.

The stiffness tensor K_{ijmn} plays the critical role of providing a linear relationship between stress and strain. The entries of K_{ijmn} , i.e. the elastic moduli, typically follow from an elastic free energy density, which for small deformations about a stable configuration is approximately quadratic in the strains:

$$f = \frac{1}{2} C_{ijmn} u_{ij} u_{mn}. \quad (2)$$

Notice that a variation of f with respect to the displacement field reveals $K_{ijmn} = \frac{1}{2}(C_{ijmn} + C_{mnij})$. Therefore, if one starts with the elastic potential energy Eq. (2), one concludes that the elastic tensor obeys the major symmetry $K_{ijmn} = K_{mnij}$.

For active solids, which exhibit energy injection at the particle or bond level, an elastic free energy is not generally well defined. Nonetheless, the internal forces still exist and can be linearized for small strains, so Hooke's law in Eq. (1) is still a valid starting place. In this case, however, the major symmetry of the elastic tensor may no longer be assumed, and instead one must consider an elastic tensor which is the sum of two parts:

$$K_{ijmn} = K_{ijmn}^e + K_{ijmn}^o, \quad (3)$$

where K_{ijmn}^e obeys the major symmetry and $K_{ijmn}^o = -K_{mnij}^o$ is antisymmetric across the major axis. Elasticity in the presence of this antisymmetric contribution K_{ijmn}^o is referred to as odd elasticity.

The existence of a nonzero anti-symmetric contribution K_{ijmn}^o implies that the solid may be brought through a closed cycle in strain space over-which energy is extracted from the solid (or injected if the cycle is run in reverse). The work per unit volume done by a solid per unit volume in an infinitesimal deformation du_{ij} is given by $dw = -\sigma_{ij} du_{ij}$. Therefore, when a patch of material is brought through a cycle in strain space which bounds the region R , the work done by

the material is:

$$\begin{aligned}
w &= - \oint_{\partial R} \sigma_{ij} du_{ij} \\
&= - \oint_{\partial R} K_{ijmn} u_{mn} du_{ij} \\
&= - \int_R K_{ijmn} du_{mn} \wedge du_{ij} \\
&= - \int_R K_{ijmn}^o du_{mn} \wedge du_{ij},
\end{aligned}$$

where we have used Stokes theorem between the second and third line. Hence, if K_{ijmn}^o is nonzero, then there exist cycles in strain space that will extract energy from the solid. Therefore, such a term does not follow from any elastic potential energy, since the energy is not a single valued function of the strain.

The inability to derive the odd elastic stresses from a potential has important implications for the underlying dynamics. The equations of motion for an (inertial) elastic solid are:

$$\rho \ddot{u}_j = \partial_i \sigma_{ij} = K_{ijmn} \partial_i \partial_m u_n, \quad (4)$$

where ρ is the density. In absence of odd elasticity, this equation of motion may be derived from the variation of the following action:

$$S = \int d^2x dt \mathcal{L}, \quad \text{with} \quad \mathcal{L} = \frac{1}{2} \rho \dot{u}_j^2 - \frac{1}{2} K_{ijmn} \partial_i u_j \partial_m u_n \quad (5)$$

(We have restricted ourselves to two spatial dimensions, for simplicity.) Importantly, notice that an antisymmetric contribution of K_{ijmn} would vanish from Eq. (5), and therefore will not be recovered in the equations of motion when the action is varied. Furthermore, since K_{ijmn}^o does not follow from an elastic potential, any naive Lagrangian density of the form $T - V$ will not yield the complete equations of motion for odd elasticity. We therefore ask: what extension of the Lagrangian formalism can capture odd elasticity?

To approach this question, we will examine an analogous technique used to resolve a similar conundrum in QCD. We will examine whether or not the odd elastic problem has the key ingredients to apply this approach directly, or if a critical modification is required. First, however, we rephrase the current problem in a way which is potentially more simple from a technical point of view.

2.1 Microscopic model

One might ask, what is the minimal microscopic model that gives rise to odd elasticity? One often conceives of a solid as a collection of masses with two-body pairwise forces: i.e. a network of masses connected by springs. In light of the ability to extract energy locally, the interaction that gives rise to odd

elasticity must be non-conservative. Furthermore, it is the linearization of the force law about an equilibrium length which gives rise to the elasticity. For a spring of rest length ℓ , the most general linear pairwise interaction in two dimensions is simply:

$$F_i(\mathbf{x}) = -(k\delta_{ij} + k^o\epsilon_{ij})(|\mathbf{x}| - \ell)\hat{x}_j, \quad (6)$$

where F_i is the force on a particle at position x_i attached to a spring with the other end of the spring affixed to the origin. In the appendix, we show an example of a system which yields this force law. If $k^o = 0$, then we recover a normal Hookean spring. Notice that $\nabla \times \mathbf{F} = k^o$, so when $k^o \neq 0$, the force is non-conservative and therefore may not be expressed as the gradient of a potential. The odd spring constant k^o sets the strength of a force which pushes perpendicular to the bond.

In reference [5], it is shown that such springs, when connected in a network, give rise to a 2D material which exhibits odd elasticity. Thus, a Lagrangian description of a single active spring would likely give rise to a Lagrangian framework for the continuum odd elastic solids. Hence, examining this model is potentially a way to reduce a field theoretic problem into a problem of a single point-like particle.

The equations of motion for the “active spring” and mass system are:

$$m\ddot{x}_i = F_i(\mathbf{x}) = -(k\delta_{ij} + k^o\epsilon_{ij})(|\mathbf{x}| - \ell)\hat{x}_j, \quad (7)$$

where m is the mass of the point particle. When $k^o = 0$, the equation of motion follows from the variation of the following action:

$$S = \int dt \mathcal{L}, \quad \text{with} \quad \mathcal{L} = \frac{1}{2}m\dot{x}_i^2 - \frac{1}{2}k(|\mathbf{x}| - \ell)^2.$$

Notice, however, that term proportional to k^o cannot be accommodated by a Lagrangian. Lastly, we mention that there are questions of stability of the trajectories of particles in the force field Eq. (7), or of an odd elastic material. We comment on these questions in the appendix.

3 An analogous problem in QCD

To gain insight into the problem of active matter, we now consider a problem explored by Witten [6] in the context of QCD. As we explain below, the basic narrative closely parallels that of odd elasticity: an antisymmetric tensor is introduced in order to lift undesired symmetries at the level of the equation of motion, and does not follow from a Lagrangian density of the usual sort. Nonetheless, the desired physics in the QCD case can be captured by adding an additional term to the action which integrates over a region in field space bounded by sphere defined by spacetime.

3.1 A field theoretic problem

Witten considers a theory with $SU(3)_L \times SU(3)_R$ symmetry spontaneously broken into the diagonal $SU(3)$. The ground state can then be parameterized by a three by three unitary matrix $U(x)$ which transforms under $(A, B) \in SU(3)_L \times SU(3)_R$ as $U \rightarrow AUB^{-1}$. To study the low energy excitations of such a theory, Witten considers the action

$$S = \frac{F^2}{16} \int d^4x \text{Tr} \partial_\mu U \partial^\mu U^{-1}, \quad (8)$$

which is the unique lorentz invariant Lagrangian which obeys $SU(3)_R \times SU(3)_L$ symmetry and contains only two derivatives. We may write $U = 1 + \frac{2i}{F} \lambda^a \pi^a$, with λ^a being the generators of $SU(3)$ normalized to $\text{Tr}(\lambda^a \lambda^b) = 2\delta^{ab}$ and π^a being the Goldstone bosons.

The reason for interest in this Lagrangian is that it contains all the symmetries of QCD, in addition to a few symmetries not present in QCD. The symmetries obeyed by the Lagrangian in Eq. (8) include:

1. Charge Conjugation: $\pi^0 \leftrightarrow \pi^0$, $\pi^+ \leftrightarrow \pi^-$, which corresponds to $U \leftrightarrow U^T$.
2. Naive Parity (P_0): $t \leftrightarrow t$, $x \leftrightarrow -x$, $U \leftrightarrow U$.
3. Boson number mod 2 ($(-1)^{N_B}$): $\pi^a \leftrightarrow -\pi^a$, or equivalently $U \leftrightarrow U^{-1}$.

The intriguing feature is that QCD in Nature only obeys the joint operation $P_0(-1)^{N_B}$, but neither P_0 nor $(-1)^{N_B}$ separately. Thus, we seek to add a term to Eq. (8) which violates P_0 and $(-1)^{N_B}$ separately but maintains $P_0(-1)^{N_B}$. The approach taken by Witten is to examine the equations of motion:

$$\frac{1}{8} F^2 \partial_\mu (U^{-1} \partial^\mu U) = 0., \quad (9)$$

and add a term that breaks the desired symmetries. There is a unique P_0 violating term with only four derivatives, and we add this term to Eq. (9) and obtain:

$$\frac{1}{8} F^2 \partial_\mu (U^{-1} \partial^\mu U) + \lambda \epsilon^{\mu\nu\alpha\beta} U^{-1} (\partial_\mu U) U^{-1} (\partial_\nu U) U^{-1} (\partial_\alpha U) U^{-1} (\partial_\beta U) = 0, \quad (10)$$

where λ is a constant. Notice that Eq. (10) still obeys $P_0(-1)^{N_B}$, but not P_0 or $(-1)^{N_B}$ separately.

Yet, there doesn't appear to be a Lagrangian which yields this equation of motion. For example, the only dimension 4 pseudo scalar is

$$\epsilon^{\mu\nu\alpha\beta} \text{Tr} [U^{-1} (\partial_\mu U) U^{-1} (\partial_\nu U) U^{-1} (\partial_\alpha U) U^{-1} (\partial_\beta U)],$$

which is zero by the anti-symmetry of $\epsilon^{\mu\nu\alpha\beta}$ and the cyclicity of the trace.

Hence, there is a strong analogy between Odd Elasticity and this Eq. (10). In both cases, a symmetry is removed at the level of the equations of motion, and the new term cannot be derived from a Lagrangian due to antisymmetry constraints. We seek to uncover the mathematical framework necessary to resolve this issue.

3.2 A toy model

As with odd elasticity, insight can be gained by considering a point particle rather than a field. We consider the case of a particle constrained to move on the surface of a two-sphere of radius 1 (embedded in 3D). The equation of motion is:

$$m\ddot{x}_i + mx_i(\dot{x}_k\dot{x}_k) = 0, \quad (11)$$

which follows from the action

$$S_0 = \int dt \frac{1}{2} m \dot{x}^2 \quad (12)$$

with a Lagrange multiplier term used to enforce the constraint that the trajectory lay on the unit sphere.

This equation of motion obeys $t \leftrightarrow -t$ and $x \leftrightarrow -x$ separately. However, we can add a term to the equation of motion which makes the system only invariant under the combination of these two operations, not each separately:

$$m\ddot{x}_i + mx_i(\dot{x}_k\dot{x}_k) = \alpha \epsilon_{ijk} x_j \dot{x}_k. \quad (13)$$

Notice that the additional term in Eq. (13) does not follow from a simple modification to the Lagrangian because the candidate term $x_i x_j \dot{x}_k \epsilon_{ijk}$ vanishes.

We see in this case the same paradigm emerging: a broken symmetry at the level of the equations of motion that gives rise to non-integrable terms in the Lagrangian.

3.3 Resolution for the toy model

We have now hit the turning point in the story, where a possible framework begins to emerge. We recognize the troublesome term in Eq. (13) as the magnetic force associated with a magnetic field $B_i = x_i/|x|^3$. Therefore, if one defines a vector potential A_i via the equation $\nabla \times \mathbf{A} = \mathbf{B}$, then the following action suffices:

$$S = \int dt \left(\frac{1}{2} m \dot{x}_i^2 + \alpha A_i \dot{x}_i \right). \quad (14)$$

However, the action in Eq. (14) seems defective since no A_i satisfying $\epsilon_{ijk} \partial_j A_k = x_i/|x|^3$ may be smoothly defined on the sphere.

We can get around this defectiveness in the context of quantum mechanical calculations. Specifically, we will consider:

$$\text{Tr } e^{iS} \equiv \int \mathcal{D}[x] e^{iS[x]}, \quad (15)$$

where the path integral goes over all periodic orbits $\gamma(t)$. In the path integral, the term of interest may be written as:

$$\exp\left(i\alpha \int dt A_i \dot{x}_i\right) = \exp\left(i\alpha \int_{\gamma} A_i dx_i\right) = \exp\left(i\alpha \int_D F_{ij} d\Sigma^{ij}\right),$$

where $\partial D = \gamma$ is the region enclosed by γ in a right handed manner with normal pointing away from the origin. Thus, if we write $\Gamma \equiv i\alpha \int_D F_{ij} d\Sigma^{ij}$, our new action is:

$$S[\gamma] = S_0 + \Gamma.$$

However, this is not the end of the story. Notice that in Eq. (15), for every $\gamma(t)$, a $\gamma(-t)$ appears and the region D' defined by $\gamma(-t)$ is the complement of D on the sphere. These two contributions must be weighted equally, so we find:

$$i\alpha \int_{D'} F_{ij} d\Sigma^{ij} = -i\alpha \int_D F_{ij} d\Sigma^{ij} + 2\pi n \quad (16)$$

$$\implies i\alpha \underbrace{\int_{D+D'} F_{ij} d\Sigma^{ij}}_{4\pi} = 2\pi n \quad (17)$$

$$\implies \alpha = \frac{n}{2}. \quad (18)$$

Thus, if the procedure we apply here is to work consistently, we need to require α to assume quantized values. This is the Dirac quantization condition.

We are ultimately interested in a problem of classical fields, i.e. elasticity. Therefore, an important question is: where exactly did the quantum mechanics come in, and how would the classical case differ? The quantum mechanics came in when we equated two expressions for the action that differ by 2π : $\Gamma \sim \Gamma + 2\pi n$. In quantum mechanics, this makes sense since the action enters into an exponential. Classically, on the other hand, we would have to require that $\Gamma_D = \Gamma_{D'}$, which implies via Eq. (17) that $\alpha = 0$ or $F_{ij} = 0$. Therefore, there can be no magnetic monopole classically, or the charged particle simply cannot couple to it. The ability to equate values of the action modulo 2π is a strictly quantum-mechanical privilege. Nonetheless, we will press onward, and we will in fact see that we encounter subtleties even before the need for quantum mechanics is invoked.

3.4 Application to QCD

Now let us map a similar structure back onto the problem encountered in QCD. In this case, instead of a time parameter t , the action involves an integral over

space-time coordinates x^μ . Instead of requiring that the trajectory of a single particle to be periodic to close the integral, we will assume that space time is a large four sphere. For the toy model, the time coordinate was mapped into the twosphere, and for the QCD problem, the space time coordinate will be mapped into the $SU(3)$ manifold.

Now we need some facts from topology. In the toy model, the time-coordinate was a 1 sphere being mapped into the two sphere. Since $\pi_1(S^2) = 0$, the trajectory of the particle enclosed a disk on the two sphere. Similarly, in the QCD problem, the four sphere traces out a 4D sub-manifold of $SU(3)$, and since $\pi_4(SU(3)) = 0$, this four sphere sub-manifold is the the boundary of a larger 5-disk. We want to express the new term Γ in the generalized action as the flux of a 5-form through this 5-disk, which is the analogy of $F_{ij}d\Sigma^{ij}$ in the toy model. This five form must be invariant under $SU(3)_L \times SU(3)_R$, and there is only one such tensor:

$$\omega_{ijklm} = -\frac{i\alpha}{240\pi^2} \text{Tr} \left[U^{-1} \frac{\partial U}{\partial y^i} U^{-1} \frac{\partial U}{\partial y^j} U^{-1} \frac{\partial U}{\partial y^k} U^{-1} \frac{\partial U}{\partial y^l} U^{-1} \frac{\partial U}{\partial y^m} \right],$$

where the y_i are coordinates on the for the bounded 5-disk in the manifold of $SU(3)$, and α is some normilization constant. Thus, the term we want to add to Eq. (8) is:

$$\Gamma = \int_Q \omega_{ijklm} d\Sigma^{ijklm} \quad (19)$$

where Q is the disk bounded by $U(x^\mu)$.

Notice furthermore that there is a quantization condition analogous the the Dirac charge quantization. Every disk Q comes with a partner Q' which is bounded by the same 4 sphere, but with an opposite orientation. This implies:

$$\int \omega_{ijklm} d\Sigma^{ijklm} = 2\pi n \quad (20)$$

for an integer n . Thus, the constant α must be quantized. Note that $\pi_5(SU(3)) = \mathbb{Z}$, so any five sphere in $SU(3)$ is simply a multiple of a basic five sphere S_0 . Setting $\alpha = 1$ is equivalent to the normalization condition $\int_{S_0} \omega d\Sigma = 2\pi$.

The action in Eq. (19) is a special case of the Wess-Zumino action. Like the additional piece added in magnetic mono-pole toy model, this is truly a quantum-mechanical structure because the value of the action is unique only modulo 2π . This ultimately allows for a nonzero coupling.

Finally, we check that the added term Γ actually produces the desired addition in Eq. (10). We can verify this at least to leading order. Using $U = 1 + \frac{2i}{F} H$,

we see that $U^{-1}\partial_i U = \frac{2i}{F}H + \mathcal{O}(H^2)$. Therefore,

$$\Gamma = \frac{2}{15\pi^2 F^5} \int_Q d\Sigma^{ijklm} \text{Tr}[\partial_i H \partial_j H \partial_k H \partial_l H \partial_m H] + \mathcal{O}(H^6) \quad (21)$$

$$= \frac{2}{15\pi^2 F^5} \int_Q d\Sigma^{ijklm} \partial_i \text{Tr}[H \partial_j H \partial_k H \partial_l H \partial_m H] + \mathcal{O}(H^6) \quad (22)$$

$$= \frac{2}{15\pi^2 F^5} \int d^4x \epsilon^{\mu\nu\alpha\beta} \text{Tr}[H \partial_\mu H \partial_\nu H \partial_\alpha H \partial_\beta H] + \mathcal{O}(H^6), \quad (23)$$

where we have used the fact that the boundary of Q is the four sphere defined by spacetime. A variation of this lagrangian yields the extra term in Eq. (10) to leading order in H .

4 Application to active solids

We saw in section four that examples from E& M and QCD bear a striking similarity to the problem at hand with active solids. In both cases, there is a symmetry broken at the level of the equations of motion which lead to a term that cannot be naively incorporated into the Lagrangian. Nonetheless, an object Γ may be added to the action whose variation results in the desired terms.

In the following sections, we wish to explore how these techniques can be applied to think about active solids. This is an open question, and in this section we will be able to show how the naive application of the preceding techniques does not fully capture the desired physics. Nonetheless, perhaps with further modifications, a suitable technique could be constructed.

4.1 Microscopic model

We will begin with the microscopic ‘‘active spring model’’ proposed above. This model is similar to the electromagnetic toy model used by Witten in that the system is described by a time parameter mapped into a two dimensional space: a two sphere in the case of the toy model, and the plane in the case of the active spring. However, an obvious physical discrepancy appears immediately. The treatment in the toy model required that the trajectory closed in order to bound a two disk. However, the trajectories implied by Eq. (7) do not close. Nonetheless, we press onward: even if the closed loops are a fiction, perhaps the equations they yield upon variation are still physical.

Since the trajectory of the particle traces out a 1 sphere in this case, we seek a two form which we can integrate over the region bounded by the trajectory. By the hodge dual, we can associate this two form with a zero form ω , and thus the term we wish to add to the action is of the form:

$$\Gamma = k^o \int_D \omega d^2x,$$

where D is the region bounded by the trajectory $\gamma(t)$, and we have anticipated the dependence on the coefficient k^o . The question at this point is: what is ω ?

We know that ω must have units of time. Indeed, this observation highlights the fact that the desired addition to the equation for the active spring is $\sim \epsilon_{zjk}x_jx_k$ as opposed to $\sim \epsilon_{zjk}x_j\dot{x}_k$, which would make the problem equivalent to that of electromagnetism.

To form an objection with dimensions of time, one might consider something of the variety $(|x| - l)/|\dot{x}|$. Yet, this option seems as though it would lead to pathological singularities, and most likely an undesirable modification to the kinetic term. Another tempting option is simply an explicit dependence on time $\omega \propto t$. This seems natural because the active spring system does not conserve energy, so one might expect explicit time dependence to appear in the action, if an action can be constructed. At first glance, a function $t(\mathbf{x})$ seems poorly defined in the interior of the region D . However, the variation of the action would only modify the boundary of D at which we could reasonably define $t = \gamma^{-1}(\mathbf{x})$. While the assignment $\omega \propto t$ does not yield the correct equation of motion, it nonetheless appears to be an intriguing direction.

4.2 Continuum elasticity

When we explore applying the quantum techniques to elasticity, new intriguing structural questions arise. The Lagrangian density for 2D elasticity involves a two component field (i.e. displacement) parameterized by a time coordinate and a 2D space coordinate. If we compactify the time-coordinate, we will likely encounter structures that would have arising from coarse graining the metabeam discussion in the preceding section. So instead, in this section, we seek to explore space coordinates behavior (which is also a natural choice since we only want to modify the stresses, but not the kinetics of the elasticity).

We therefore take our material to be a topological sphere (by insisting that the strains vanish at infinity). Notice that the displacement field takes values in \mathbb{R}^2 . This causes our first subtlety: it doesn't make much sense for the two sphere to define the boundary of a three dimensional region in \mathbb{R}^2 . This did not arise for the QCD case, since spacetime is 4 dimensional and $SU(3)$ is 8 dimensional. We are not out of luck yet, though. We can introduce degrees of freedom to allow a field space with high enough dimension for the techniques to be plausible.

To do this, we take a derivative of the equations of motion Eq. (4) to obtain:

$$\rho\ddot{u}_{jk} = K_{ijmn}\partial_i\partial_m u_{kn}.$$

When K_{ijmn} has the major symmetry, this equation of motion follows from the lagrangian density:

$$\mathcal{L} = \frac{1}{2}\rho\dot{u}_{ij}^2 + K_{ijmn}\partial_i u_{jk}\partial_m u_{nk}. \quad (24)$$

The displacement gradient tensor lives in \mathbb{R}^4 . Therefore, the map from $S^2 \rightarrow \mathbb{R}^4$ defines a two dimensional surface which bounds a three dimensional ball Q . We

wish to evaluate a three form over this ball. This three form will be dual to a one form, so we may write:

$$\Gamma = \int_Q dn^\alpha \omega_\alpha$$

where n_α is dual to the differential 3 form on Q . Here, we are using the Greek index to refer to a given basis for the strain. (For isotropic solids, a useful basis is τ^α in reference [5].) In this basis, K_{ijmn}^o may be written as a four by four matrix $K_{\alpha\beta}^o$. A natural guess would be that $\omega_\alpha \propto K_{\alpha\beta}^o U_\beta$. Unfortunately, such a term does not contribute properly to the equations of motion, since it can actually be written as a term in the normal lagrangian:

$$\begin{aligned} \int_Q dn^\alpha K_{\alpha\beta}^o U_\beta &= \int_{\partial Q} K_{\alpha\beta}^o dU^\alpha \wedge dU^\beta \\ &= \int_{S^2} K_{\alpha\beta}^o \epsilon_{ij} \partial_i U^\alpha \partial_j U^\beta d^2x. \end{aligned}$$

A variation gives a term $K_{\alpha\beta}^o \epsilon_{ij} \partial_i \partial_j U^\beta = 0$, and does not contribute to the equations of motion. Nonetheless, the approach seems quite close, yet the naive applications of the quantum techniques do not play out as expected. It is still an open question whether this general approach to generalizing the action will always fail, or if there is a slight modification that would capture odd elasticity. Indeed this is an intriguing area for additional inquiry.

5 Conclusion

We have explicated parallels between problems arising in the Lagrangian formalism of classical, active solids, and quantum mechanical fields. In the context of QCD and electromagnetism, certain symmetry violating terms in the equations of motion could be accounted for by adding a topological term to the action. Straight-forward attempts to apply this technique to odd elasticity encounter various complications, but nonetheless provide a stimulating source of potential questions and ideas.

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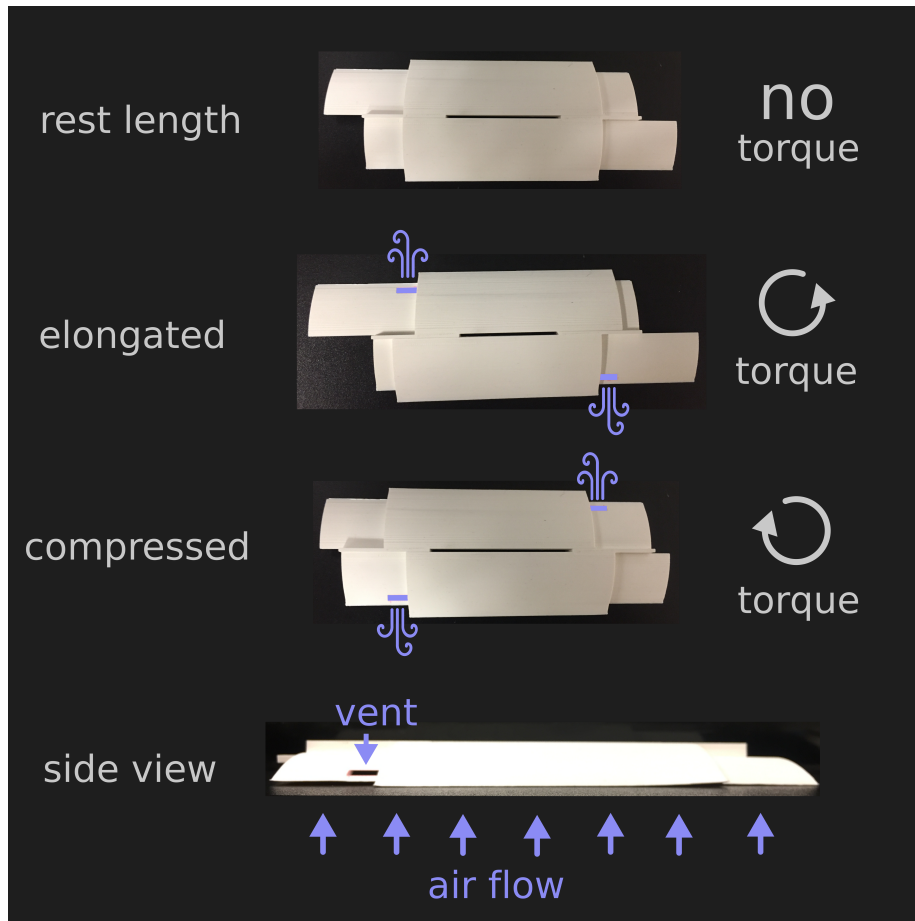


Figure 1: An experimental realization of an active metabeam.

6 Appendix

6.1 Possible Experimental Realization

Fig (1) represents a possible experimental realization of the force law in Eq. (6). A 3D printed puck sits on an air hockey table. When the puck is elongated, the vents open with a given chirality, yielding a torque. When the puck is contracted, the vents open with the opposite chirality. The background airflow is integrated out, yielding an “active” system which does not conserve energy. Such beams would be connected in a network to form an odd elastic solid. A spring could be added to provide the standard Hookean restoring force.

6.2 Stability

In reference [5], it is shown in detail how to compute the spectrum of solids with odd elasticity. One finds that an inertial equation of motion $\rho\ddot{u}_j = \partial_i\sigma_{ij}$ is dynamically unstable for a wide range of possible K_{ijmn}^o . Furthermore, an inertial equation of motion is rather un-physical when discussing active systems, which often exist in contact with a substrate or immersed in a low Reynolds number fluid. Therefore, in practice, the simplest realistic model is that of an overdamped solid with $\eta\dot{u}_j = \partial_i\sigma_{ij}$, where η is a drag coefficient. In the context of a single active spring, we have $\gamma\dot{x}_i = F_i$, with γ a drag coefficient. These terms are also intriguing from the point of view of variational mechanics because they represent strictly dissipative frictional forces. However, the modifications do not affect the antisymmetric force terms which are the focus of this work, and are often incorporated via a Rayleigh dissipation functional.