Problem Set 3

Physics 445

Due April 29

Some abbreviations: P&S - Peskin & Schroeder

1. Work out the Poincaré algebra by computing the commutators of the generators $M_{\mu\nu}$ and P_λ. You can do this using any convenient representation of the generators. As a matter of nomenclature, the commutation relations of the $M_{\mu\nu}$ alone define the Lorentz algebra.

2. Now let's explore spinor representations in more detail. The Dirac matrices are a representation of a Clifford algebra with defining relations:

$$\{\Gamma^\mu, \Gamma^\nu\} = \eta^{\mu\nu}. \tag{0.1}$$

Let us take the mostly minus signature we used in lecture. The curly bracket denotes the anticommutator $\{a, b\} = ab + ba$.

The minimal size matrices that solve the Clifford algebra relation (0.1) are $2^{[d/2]} \times 2^{[d/2]}$ where $[\ldots]$ denotes least integer and d is the space-time dimension. This is why the Dirac matrices are 4×4 in $d = 4$. The spinors on which the Dirac matrices act define the Dirac representation.

(i) Show that the Lorentz generators $\Sigma^{\mu\nu} = \frac{i}{4}[\Gamma^\mu, \Gamma^\nu]$ satisfy the Lorentz algebra you found in problem 1.

(ii) In even dimensions, the Dirac representation is always reducible. Define $\Gamma^5 = i\Gamma^0 \cdots \Gamma^3$. Check that Γ^5 anticommutes with all Γ^μ and the Lorentz generators $\Sigma^{\mu\nu}$. Also check that $(\Gamma^5)^2 = 1$. We can therefore reduce the Dirac representation to two Weyl representations with eigenvalues ± 1 under Γ^5, and this decomposition is Lorentz invariant.

(iii) The Minkowski metric is real. So we can complex conjugate (0.1) and conclude that $\Gamma^{\mu*}$ and $-\Gamma^{\mu*}$ also satisfy the same Clifford algebra relation. They should be related to Γ^μ by a change of basis. We can find this change of basis for a specific presentation of the Dirac matrices like the Weyl basis given in lecture. In that basis, Γ^2 is imaginary and the other matrices are real.

We can define two combinations: $B_1 = \Gamma^2$ and $B_2 = \Gamma^5\Gamma^2$. Check that

$$B_1\Gamma^\mu B_1^{-1} = -\Gamma^{\mu*}, \quad B_2\Gamma^\mu B_2^{-1} = \Gamma^{\mu*}, \quad B_i\Gamma^5 B_i^{-1} = -\Gamma^{5*},$$

and that

$$B_i \Sigma^{\mu\nu} B_i^{-1} = \Sigma^{\mu\nu *},$$

for $i = 1, 2$. The B_i are the only two matrices with this action on $\Sigma^{\mu\nu}$.

The action on the Lorentz generators implies that a Dirac spinor ψ and $B^{-1}\psi^*$ transform the same way under Lorentz. So the complex conjugate of a Weyl spinor in $d = 4$ has the opposite chirality under Γ^5. Now we can discuss imposing a reality condition on the spinor ψ. The reality condition must make sense in any frame so both sides must transform the same way under Lorentz:

$$\psi^* = B_i \psi.$$

If we complex conjugate this condition, it is only consistent if $B_i^* B_i = 1$. In $d = 4$, this is not true for B_2 but it is true for B_1. We can impose a Majorana condition on a Weyl spinor only if the Weyl spinor is conjugate to itself. This is not true in $d = 4$ as we saw above but it is true in $d = 2 \bmod 8$.

(iv) In $d = 4$, we can only have either a Majorana condition or a Weyl condition. Classically, it must be the case that we can map a Weyl spinor to a Majorana spinor and vice-versa. A Weyl spinor η with definite chirality maps to a Majorana spinor via:

$$\eta \to \eta + B_1^* \eta^*.$$

A Majorana ζ maps to a Weyl via:

$$\zeta \to P_+ \zeta$$

using the projection operator $P_+ = \frac{1}{2}(1 + \Gamma^5)$ onto definite chirality. Show that these maps are inverses.