# Chern Simons Theory and Rational CFT on Manifolds with Boundary

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Abstract: We explore the connection between Chern Simons theory on a manifold and rational conformal field theories (CFTs) defined on the boundary of the manifold, by mapping the degrees of freedom of the two theories and defining the Hilbert space of the bulk Chern Simons theory through this map. The correspondence holds both with and without sources in the bulk on the manifold, although the existence of sources in the bulk modifies the dual boundary CFT and hence also the Hilbert space.

# Contents



## 1 Introduction

Naively, the dynamics of Chern Simons theory on an arbitrary manifold $M$ seem to be trivial, since the action of the theory

$$S = \frac{k}{4\pi} \int_M \text{Tr}\left(AdA + \frac{2}{3}A^3\right), \tag{1.1}$$

is purely topological, implying that the stress tensor vanishes, and so does the Hamiltonian. However, the quantization of the theory turns out to be subtle on manifolds with boundary and on nontrivial compact manifolds, wherein dynamics can be generated from pure gauge degrees of freedom residing on the boundary of the manifold and/or those with nontrivial holonomies. In this review, we shall focus on the first case of manifolds with boundaries and explore how nontrivial dynamics are generated by 'large gauge transformations', i.e. gauge transformations that don't die down to unity on the boundary.

In the cases where $M = \Sigma \times \mathbb{R}$, where $\Sigma$ is a two dimensional manifold with boundary $\partial\Sigma \neq \emptyset$, one can interpret the real line as time and canonically quantize the theory on a spatial slice $\Sigma$. Canonical quantization of theories with constraints can be done in one of two ways, by either constraining the theory first and then quantizing or by imposing the constraints as operator equations on the Hilbert space of the theory. Both approaches reveal close connections between Chern Simons theory and rational CFTs known as Wess-Zumino-Witten (WZW) models. The first approach reveals a direct equivalence between the actions of the two theories, mapping the degrees of freedom on both sides. The second approach demonstrates an equivalence of the Hilbert space of Chern Simons theory and the space of conformal blocks of WZW models.

Chern Simons theory was originally developed by Witten in his seminal work on the Jones polynomial in knot theory[1]. The arguments of this paper were subsequently made concrete in [2–5]. The intimate connection between Chern Simons theory and rational CFT was also explored in [1, 2, 6].

This review is organized as follows. In section 1, we begin with a historic introduction to WZW models, while section 2 is devoted to a few examples of canonical quantization of Chern Simons theory, where we will explicitly work out the statements above.

## 2 Wess Zumino Witten Models

WZW models were originally introduced by Witten in the context of non-abelian bosonization of fermions in two dimensions[7]. Free massless fermions in two dimensions are conformally invariant with two conserved currents (the chiral components of the vector current). Hence, the corresponding bosonic theory must also have the same two properties. Since $N$ Dirac fermions have a $U(N)_L \times U(N)_R$ global symmetry, it is natural to express them in terms of a bosonic field $g$ that takes values in $U(N)$, with the global symmetry acting on $g$ by $g \to \Omega g \tilde{\Omega}^{-1}$ for $\Omega, \tilde{\Omega} \in U(N)$. Now, observe that if we write

$$J_+ = \frac{i}{2\pi} g^{-1}\partial_+ g, \qquad J_- = -\frac{i}{2\pi}\left(\partial_- g\right) g^{-1}, \tag{2.1}$$

in lightcone coordinates, separate conservation equations for both currents, $\partial_- J_+ = 0 = \partial_+ J_-$ are not only compatible with each other, but also equivalent[1].

Next, we ask the question: what action would result in the currents above? The obvious guess would be the non-linear sigma model

$$S = \int \frac{1}{4\lambda^2} \operatorname{Tr}\left(\partial_\mu g \partial^\mu g^{-1}\right). \tag{2.2}$$

However, it is well known that the non-linear sigma model (2.2) is asymptotically free with the interactions getting stronger in the IR. Hence, it can't be dual to a conformal field theory. Secondly, it also results in fewer conserved currents, namely

$$\partial_\mu \left(g^{-1}\partial_\mu g\right) = 0. \tag{2.3}$$

It turns out that the non-linear sigma model can be modified in order to obtain a conformally invariant theory with the correct conserved currents as follows. Consider, instead of $SU(N)$, an arbitrary symmetry group $G$ such that the bosonic field $g$ now takes values in $G$[2]. Next, pick a manifold $B$ such that its boundary $\partial B$ is the compactification of our

[1]The expressions (2.1) are obtained from a generalization of the corresponding expressions in the abelian case [8, 9]. In the non-abelian case, however, there is an ordering ambiguity that is resolved by picking the ordering in the above expressions.

[2]The dual free massless fermions will now take values in a $G$ multiplet.

original spacetime. Every map $g : \partial B \to G$ can be extended into a map $\tilde{g} : B \to G$, and we can construct the Wess-Zumino term

$$\Gamma = \frac{1}{24\pi}\int_B d^3y\ \epsilon^{\alpha\beta\gamma}\,\mathrm{Tr}\left(\tilde{g}^{-1}\partial_\alpha\tilde{g}\tilde{g}^{-1}\partial_\beta\tilde{g}\tilde{g}^{-1}\partial_\gamma\tilde{g}\right). \tag{2.4}$$

The extension $\tilde{g}$ is not unique, and it can be shown that consequently the Wess-Zumino term is well defined modulo $\Gamma \to \Gamma + 2\pi$. It can also be shown that the integral (2.4) can be written as an integral over spacetime $\partial B$, with the explicit expression depending on the group $G$. Hence, we can consider the action

$$S = \frac{1}{4\lambda^2}\int d^2x\ \ \mathrm{Tr}\,\partial_\mu g\partial^\mu g^{-1} + k\Gamma' \tag{2.5}$$

with $k \in \mathbb{Z}$, so that the path integral is well defined. This is the Wess-Zumino-Witten model. Working out the equations of motion results in the expression

$$\left(\frac{1}{2\lambda^2}+\frac{k}{8\pi}\right)\partial_-(g^{-1}\partial_+g) + \left(\frac{1}{2\lambda^2}-\frac{k}{8\pi}\right)\partial_+(g^{-1}\partial_-g) = 0. \tag{2.6}$$

We see that at $\lambda^2 = |4\pi/k|$, we obtain the desired conserved currents (or their parity conjugates). Note that at this value of the coupling constant, the theory is invariant under the transformation

$$g \to \Omega(x^+)g\tilde{\Omega}(x^-), \tag{2.7}$$

i.e., we have promoted the global $G \times G$ symmetry to a gauge symmetry. It can also be shown that the WZW model is conformally invariant whenever $\lambda^2 = |4\pi/k|$, by showing that the beta function vanishes.

Note that the general solution to the conservation equations is $g(x^+, x^-) = A(x^-)B(x^+)$ for $k > 0$, which implies that left and right moving degrees of freedom decouple, exactly as in the case of free fermions, leading to the conjecture that WZW models are dual to free fermions.

## 2.1 Quantization

Quantizing the theory amounts to computing the Poisson brackets of the degrees of freedom (in this case, the conserved currents) and promoting them to commutation relations. The Hilbert space then lies in a representation of the current algebra[3].

We work in light-cone coordinates by treating $\sigma = x^-$ as space and $\tau = x^+$ as time. The action then becomes first order in time derivatives. However, this procedure fails to give the Poisson brackets of operators containing $\tau$ derivatives, since any initial data on $\tau = 0$ may not be able to completely predict all degrees of freedom in the future, such as the left-moving degrees of freedom on a $\tau =$ constant frame. In order to obtain the full current algebra, we must also obtain the commutation relations of the left-moving degrees of freedom by treating $x^-$ as time and $x^+$ as space.

[3]Since the symmetry is now a gauge symmetry, the Hilbert space is not required to be a singlet under gauge transformations.

The action in light cone coordinates is given by

$$S = \frac{k}{16\pi} \int d\sigma d\tau \, \mathrm{Tr} \, \partial_\tau g \partial_\sigma g^{-1} + k\Gamma, \tag{2.8}$$

where $\Gamma$ is also first order in time derivatives. Note that this action is not chirally invariant, but since it only accounts for the right-miving degrees of freedom, we don't expect it to preserve chirality. The theory (2.8) on its own is known as the Chiral Wess-Zumino-Witten (CWZW) model, and we'll encounter it later in the next section.

Note that (2.8) is already in Hamiltonian form, since it is first order in time derivatives. This makes it difficult to split the degrees of freedom into coordinates and momenta and we must use alternate methods to compute Poisson brackets. For a theory with dynamical variables $\phi^i$ and an action of the form

$$S = \int dt A_i(\phi) \frac{d\phi^i}{dt}, \tag{2.9}$$

an infinitesimal change in variables $\phi^i \to \phi^i + \delta\phi^i$ results in a change in the action of the form

$$\begin{aligned} \delta S &= \int dt \left( \frac{\partial A_i}{\partial \phi^j} \delta\phi^j \frac{d\phi^i}{dt} + A_i \frac{d}{dt} \delta\phi^i \right), \\ &= \int dt \left( \frac{\partial A_j}{\partial \phi^i} - \frac{\partial A_i}{\partial \phi^j} \right) \delta\phi^i \frac{d\phi^j}{dt}, \\ &= \int dt \; F_{ij} \delta\phi^i \frac{d\phi^j}{dt}. \end{aligned} \tag{2.10}$$

Define $F^{jk}$ as the inverse of the matrix $F_{ij}$[4]. The Poisson bracket of two functions $X$ and $Y$ on phase space is then given by

$$[X, Y]_{\mathrm{PB}} = F^{ij} \frac{\partial X}{\partial \phi^i} \frac{\partial Y}{\partial \phi^j}. \tag{2.11}$$

Varying the action (2.8) gives the equation

$$\delta S = \frac{k}{4\pi} \int d\sigma d\tau \, \mathrm{Tr} \, g^{-1} \delta g \frac{\partial}{\partial \sigma} \left( g^{-1} \frac{dg}{dt} \right). \tag{2.12}$$

In order to compute the Poisson brackets, it is not necessary to choose coordinates $\phi^i$ on the phase space. All we need to do is pick a basis for the tangent vectors to the phase space, from which the matrices $F_{ij}$ and $F^{jk}$ can be constructed. We will work with the basis of matrices $g^{-1}\delta g(\sigma)$, in which $F$ acts on both the Lie algebra index of $g^{-1}\delta g(\sigma)$ and on $\sigma$. In this basis, we identify the matrix $F$ from (2.12) as

$$F = 1 \otimes \frac{k}{4\pi} \frac{\partial}{\partial \sigma}, \tag{2.13}$$

[4]The matrix $F_{ij}$ is the symplectic form on the phase space of the theory

with '1' acting on the Lie algebra index, and $(k/4\pi)\partial_\sigma$ acting on the the $\sigma$ 'index'. The inverse of $F$ is then given by

$$F^{-1} = 1 \otimes \frac{4\pi}{k}\left(\frac{\partial}{\partial\sigma}\right)^{-1}. \tag{2.14}$$

Let us consider the Poisson brackets of $X = \operatorname{Tr} A\partial_\sigma g g^{-1}(\sigma)$ and $Y = \operatorname{Tr} B\partial_{\sigma'} g g^{-1}(\sigma')$ where the matrices $A$ and $B$ are generators of the group $G$. We first compute

$$\delta X \delta Y = \frac{\partial X}{\partial \phi^i}\frac{\partial Y}{\partial \phi^j}\delta\phi^i\delta\phi^j, \tag{2.15}$$

and then replace $\delta\phi^i\delta\phi^j$ by $F^{ij}$ to obtain the Poisson bracket of $X$ and $Y$. We have

$$\begin{aligned} \delta X &= \operatorname{Tr} A\partial_\sigma(\delta g)g^{-1} - \operatorname{Tr} A\partial_\sigma g g^{-1}\delta g g^{-1}, \\ &= \operatorname{Tr} g^{-1}Ag\partial_\sigma\left(g^{-1}\delta g\right). \end{aligned} \tag{2.16}$$

Similarly, we can evaluate $\delta Y$ to obtain

$$\delta X\delta Y = \operatorname{Tr} g^{-1}(\sigma)Ag(\sigma)\partial_\sigma\left(g^{-1}\delta g(\sigma)\right)\cdot \operatorname{Tr} g^{-1}(\sigma')Bg(\sigma')\partial_{\sigma'}\left(g^{-1}\delta g(\sigma')\right). \tag{2.17}$$

We have chosen the basis vectors of the cotangent space $(g^{-1}\delta g(\sigma))^a$ and $(g^{-1}\delta g(\sigma'))^b$. Hence, we must replace $(g^{-1}\delta g(\sigma))^a(g^{-1}\delta g(\sigma'))^b$ by $\delta^{ab}(4\pi/k)\theta(\sigma,\sigma')$, where $\theta(\sigma,\sigma')$ is an inverse of $\partial_\sigma$. This amounts in replacing $\partial_\sigma(g^{-1}\delta g(\sigma))^a \cdot \partial_{\sigma'}(g^{-1}\delta g(\sigma'))^b$ by $\delta^{ab}(4\pi/k)\partial_\sigma \cdot \partial_{\sigma'}\theta(\sigma,\sigma') = \delta^{ab}(4\pi/k)\delta'(\sigma-\sigma')$, resulting in the Poisson bracket

$$\begin{aligned} \left[\operatorname{Tr} AJ_-(\sigma), \operatorname{Tr} BJ_-(\sigma')\right]_{\text{PB}} &= \frac{4\pi}{k}\delta'(\sigma-\sigma')\operatorname{Tr} g^{-1}(\sigma)Ag(\sigma)g^{-1}(\sigma')Bg(\sigma'), \\ &= \frac{4\pi}{k}\delta(\sigma-\sigma')\operatorname{Tr}[A,B]\partial_\sigma g^{-1} + \frac{4\pi}{k}\delta'(\sigma-\sigma')\operatorname{Tr} AB. \end{aligned} \tag{2.18}$$

Rescaling the current as $J_- = (k/2\pi)\partial_\sigma g^{-1}$ and translating the Poisson brackets to canonical commutation relations, we obtain the current algebra for the WZW model

$$\begin{aligned} \left[\operatorname{Tr} AJ_-(x), \operatorname{Tr} BJ_-(y)\right] &= 2i\delta(x-y)\operatorname{Tr}[A,B]J_-(x) + k\frac{i}{\pi}\delta'(x-y)\operatorname{Tr} AB, \\ \left[\operatorname{Tr} AJ_+(x), \operatorname{Tr} BJ_+(y)\right] &= 2i\delta(x-y)\operatorname{Tr}[A,B]J_+(x) + k\frac{i}{\pi}\delta'(x-y)\operatorname{Tr} AB, \\ [J_-, J_+] &= 0, \end{aligned} \tag{2.19}$$

with the second equation in (2.19) obtained similarly by reversing the 'time' and 'space' coordinates.

(2.19) is known as (two copies of) the Kac-Moody algebra. Another indicator of the fact that $k$ can take only integer values is that the Kac-Moody algebra has well behaved unitary representations only when $k$ is an integer.

In two dimensional conformal field theory, it turns out to be convenient to use complex coordinates $z = x^0 + ix^1$ and $\bar{z} = x^0 - ix^1$ instead of light cone coordinates. In these coordinates, it is a general property of CFTs that the currents factorize into holomorphic and antiholomorphic parts $J(z)$ and $\bar{J}(\bar{z})$. The holomorphic and antiholomorphic currents are expressed as linear combinations of the generators of $G$ as $J(z) = J^a(z)T^a$ and $\bar{J}(\bar{z}) = \bar{J}^a(\bar{z})T^a$. The coefficients in this basis can then be expanded in a Laurent series as

$$\begin{aligned} J^a(z) &= \sum z^{-n-1} J^a_n, \\ \bar{J}^a(\bar{z}) &= \sum \bar{z}^{-n-1} \bar{J}^a_n. \end{aligned} \tag{2.20}$$

These relations can be inverted to give

$$\begin{aligned} J^a_n &= \frac{1}{2\pi i} \oint dz\ z^n J^a(z), \\ \bar{J}^a_n &= \frac{1}{2\pi i} \oint d\bar{z}\ \bar{z}^n \bar{J}^a(\bar{z}). \end{aligned} \tag{2.21}$$

The commutation relations of the modes of the currents are then given by

$$\begin{aligned} \left[J^a_n, J^b_m\right] &= \sum_c i f^{ab}_c J^c_{n+m} + kn\delta^{ab}\delta_{n+m,0}, \\ \left[\bar{J}^a_n, \bar{J}^b_m\right] &= \sum_c i f^{ab}_c \bar{J}^c_{n+m} + kn\delta^{ab}\delta_{n+m,0}, \\ \left[J^a_n, \bar{J}^b_m\right] &= 0. \end{aligned} \tag{2.22}$$

### 2.2 Representations of the Kac-Moody Algebra

The abstract way of defining the Laurent modes of the conserved currents is to take the tensor product of the simple Lie algebra $\mathfrak{g}$ corresponding to a simple Lie group $G$ and the algebra of Laurent polynomials in some variable $z$[10]. This algebra is generated by elements of them form

$$J^a_n = J^a \otimes z^n \partial_z. \tag{2.23}$$

Equation (2.21) gives an explicit realization of this tensor product. The Kac-Moody algebra is the central extension of the algebra of the modes (2.23).

We construct the Laurent modes of the Cartan subalgebra and the ladder operators $\{H^i, E^\alpha\}$ of $\mathfrak{g}$ by defining

$$\begin{aligned} H^i_n &= H^i \otimes z^n \partial_z, \\ E^\alpha_n &= E^\alpha \otimes z^n \partial_z. \end{aligned} \tag{2.24}$$

The commutation relations of these elements are given by

$$\begin{aligned} \left[H^i_n, H^i_m\right] &= kn\delta^{ij}\delta_{n+m,0}, \\ \left[H^i_n, E^\alpha_m\right] &= \alpha^i E^\alpha_{n+m}, \\ \left[E^\alpha_n, E^\beta_m\right] &= \frac{2}{\alpha \cdot \alpha}\left(\alpha \cdot H_{n+m} + kn\delta_{n+m,0}\right) && \text{if} \quad \alpha = -\beta, \\ &= \mathcal{N}_{\alpha\beta} E^{\alpha+\beta}_{n+m} && \text{if} \quad \alpha + \beta \in \Delta, \\ &= 0 && \text{otherwise,} \end{aligned} \tag{2.25}$$

where $\Delta$ is the set of all roots of $\mathfrak{g}$, and $\mathcal{N}_{\alpha\beta}$ is some normalization.

In order to distinguish the operators with different values of $n$, we introduce a grading operator $L_0$ which, under a commutator, pulls out the sum of $n$-values from a string of operators

$$\left[L_0, J^a_n J^b_m \cdots\right] = -(n+m+\cdots)\left(J^a_n J^b_m \cdots\right). \tag{2.26}$$

The Cartan subalgebra is the set of operators

$$\left\{H^1_0, \cdots, H^r_0, k, L_0\right\}. \tag{2.27}$$

Operators with positive values of $n$ will be considered as raising operators, while those with negative values will be considered as lowering operators. Ladder operators with $n = 0$ corresponding to a positive root will be added to the set of raising operators and those corresponding to a negative root will be added to the set of lowering operators.

As in the case of simple Lie algebras, we begin with a highest weight state $|\lambda\rangle$, which is a simultaneous eigenstate of the Cartan subalgebra. Conventionally, we choose the $L_0$ eigenvalue on the highest weight state to be 0. The highest weight state is annihilated by all raising operators.

$$E^\alpha_0 |\lambda\rangle = E^{\pm\alpha}_n |\lambda\rangle = H^i_n |\lambda\rangle = 0, \qquad n > 0,\ \alpha > 0. \tag{2.28}$$

The remaining states in the representation are obtained by action of the the lowering operators on $|\lambda\rangle$. The set of states generated by this procedure is called a *Verma module* $V_\lambda$. We will often abuse notation and label the Verma module $V_\lambda$ by the highest weight $\lambda$. The *grade* or *level* of a state is defined to be its $L_0$ eigenvalue. The set of states at a level $N$ is spanned by a basis which contains elements of the form

$$E^{\alpha_1}_{-k_1} E^{\alpha_2}_{-k_2} \cdots |\lambda\rangle, \qquad k_i \geq 0,\ \sum k_i = N, \tag{2.29}$$

or alternately

$$J^{a_1}_{-k_1} J^{a_2}_{-k_2} \cdots |\lambda\rangle, \qquad k_i \geq 0,\ \sum k_i = N. \tag{2.30}$$

Unlike simple Lie algebras, this procedure does not result in negative norm states. Hence, the Verma module is infinite dimensional. The highest weight, however, does get restricted to discrete values through the inequality

$$k \geq (\lambda, \theta), \qquad k \in \mathbb{Z}_+, \tag{2.31}$$

where $(\lambda, \theta)$ is the inner product of the highest weight $\lambda$ of the given representation and the highest weight $\theta$ of the adjoint representation[5] (also called the highest root). It is a property of simple Lie algebras and Kac-Moody algebras that the inner product of weights with roots must be an integer. This gives a finite set of possible values of $\lambda$ for a fixed $k$.

For $k = 1$ there can only be one highest weight and hence the representation is essentially unique. This representation is known as the *basic representation.* For general values of $k$, the representations obtained are tensor products of the basic representation with different symmetry and antisymmetry conditions. The representations obtained in this manner are called *integrable representations.*

[5] Recall that in the adjoint representation, the elements of the Lie algebra act upon themselves through the Lie bracket, i.e., $T_1 |T_2\rangle = |[T_1, T_2]\rangle$. The roots of a Lie algebra are the weights of the adjoint representation.

## 3 Canonical Quantization of Chern Simons Theory

Equipped with the quantization of WZW models, we are now ready to tackle the canonical quantization of Chern Simons theory. In the rest of this section we will closely follow the procedure outlined in [2].

We begin by considering suitable boundary conditions for the action

$$S = \frac{k}{4\pi}\int_M \mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right), \tag{3.1}$$

[6]where we have suppressed the wedge products between differential forms for convenience. The variation of the action results in

$$\delta S = -\frac{k}{4\pi}\int_{\partial M} \mathrm{Tr}\left(A\delta A\right) - \frac{k}{2\pi}\int_M \mathrm{Tr}\left(F\delta A\right). \tag{3.2}$$

For the equations of motion to be local, the boundary term in (3.2) must vanish. Hence we set one of the components of the gauge connection $A$ along an arbitrary boundary direction to be zero. For manifolds of the form $M = \Sigma \times \mathbb{R}$, we interpret $\mathbb{R}$ as time and set the boundary conditions $A_0 = 0$ on $\partial\Sigma$. The symmetry of this theory can be split up into two subsets, transformations that die down to unity on $\partial\Sigma$ and transformations that don't (these have to be time independent on the boundary to be compatible with the boundary condition). Transformations that are unity on the boundary form the gauge symmetry while the rest must be considered global. Decomposing the derivatives and the gauge field into time and space components as $d = dt\partial/\partial t + \tilde{d}$ and $A = A_0 + \tilde{A}$ results in the action

$$S = -\frac{k}{4\pi}\int_M \mathrm{Tr}\left(\tilde{A}\frac{\partial \tilde{A}}{\partial t}dt\right) + \frac{k}{2\pi}\int_M \mathrm{Tr}\left[A_0\left(\tilde{d}\tilde{A} + \tilde{A}^2\right)\right], \tag{3.3}$$

plus a boundary term that vanishes under our boundary condition. We can see from this expression that $A_0$ is a Lagrange multiplier that imposes the constraint $\tilde{F} = \tilde{d}\tilde{A} + \tilde{A}^2 = 0$, i.e., ensures that the magnetic field vanishes. In other words, the degrees of freedom of the theory are flat connections on $\Sigma$. Plugging in the general solution to this constraint back into the action results in an effective action for Chern Simons theory that gets rid of the gauge ambiguity[7].

Next, we work out this procedure explicitly for a few examples.

### 3.1 $\Sigma = D$, a Disk Centered at the Origin

Since the interior of the disk is topologically trivial, the constraint is solved by

$$\tilde{A} = UdU^{-1}, \tag{3.4}$$

where $U$ is a single valued map from $D \times \mathbb{R}$ to $G$. The change of variables produces a unit Jacobian and we obtain the effective action (see appendix A for a detailed derivation)

$$S = kS_C^{+}(U) \equiv \frac{k}{4\pi}\int_{\partial M} \mathrm{Tr}\left(U^{-1}\partial_\phi U U^{-1}\partial_t U\right) + \frac{k}{12\pi}\int_Y \mathrm{Tr}\left(U^{-1}dU\right)^3, \tag{3.5}$$

[6]For arbitrary gauge groups $G \neq SU(N)$, the trace should be replaced by the Killing form $\langle\ ,\ \rangle$.

[7]The resulting effective theory could still be 'gauge' invariant since we haven't dealt with glocal transformations yet, as we will shortly see.

which is identical to the action of the chiral Wess-Zumino-Witten model (2.8) with the 'space' coordinate compactified into a circle, and depends only on the boundary value of $U$. This is because the value of $U$ in the bulk can always be changed by a bulk gauge transformation. The effective action is invariant under transformations of the kind

$$U \to \tilde{V}(\phi) U V(t), \tag{3.6}$$

where the $\tilde{V}(\phi)$ accounts for the global symmetry of Chern Simons theory, and $V(t)$ reflects a redundancy in the parametrization of $A$ by $U$. The phase space of the theory is hence the set of based loops in the group $LG/G$, where $LG$ is the loop group, i.e., the set of maps $S^1 \to G$. As we saw in the previous section, for actions that are first order in time derivatives of the form $S = \int \mathcal{A}_i (d\phi^i/dt) dt$, the symplectic form is given by $\omega = \delta \mathcal{A}$, where $\delta$ is the exterior (antisymmetric) derivative on the phase space. Hence, the symplectic form for Chern Simons theory on a disk is given by

$$\omega = \frac{k}{4\pi} \oint \mathrm{Tr} \left( U^{-1} \delta U \right) \frac{d}{d\phi} \left( U^{-1} \delta U \right). \tag{3.7}$$

Note that the gauge field $A_\phi$ is nothing but the Kac-Moody current for the CWZW model, and hence, the resulting Hilbert space is the trivial representation of the Kac-Moody Algebra, with the highest weight state being the state-operator dual of the identity operator, i.e., the Hilbert space is the space of descendants of the identity operator in the dual WZW model.

### 3.2 Adding Sources

We can add static sources to the theory by including Wilson lines running through the bulk, piercing $\Sigma$ at points $P_i$ in representations $\lambda_i$[8]. Since the Wilson correspond the static charges in the cylinder, the constraint equations should intuitively be modified to

$$\frac{k}{8\pi} \epsilon^{ij} F^a_{ij} = \sum_m \delta(x - P_m) T^a_{(m)}, \tag{3.8}$$

where $T^a_{(m)}$ are the generators of $\mathfrak{g}$ associated to the external charges. Classically, this equation is nonsensical. The solution to this equation will not be an ordinary $c$-number connection, since we have non-commuting operators on the right hand side. To paraphrase Witten from [1], "A representation of a group should be seen as a quantum object. This representation should be obtained by quantizing a classical theory." Hence, there must be a classical theory whose quantization would result in the representations above, and there is. We use the Borel-Weil-Bott theorem to canonically associate a representation $\lambda$ corresponding to a Wilson line to the symplectic phase space $G/T$, $T$ being the maximal torus in $G$, with the symplectic form

$$\omega = \mathrm{Tr}\, \lambda \left( g^{-1} \delta g \right)^2, \tag{3.9}$$

with $g$ a time dependent element in $G$ and $\lambda = \boldsymbol{\lambda} \cdot \mathbf{H}$ the highest weight put into the Cartan subalgebra.

[8]Recall that representations of simple groups are labeled by highest weights

This classical phase space can be obtained from the action

$$\int dt \, \mathrm{Tr} \, \lambda g^{-1}(t) \left(\partial_0 + A_0\right) g(t), \tag{3.10}$$

which is invariant under $g(t) \to g(t)h(t)$ such that $h(t) \in T$ commutes with $\lambda$. This results in the classical phase space $G/T$ with the above symplectic structure. Hence we have total effective action

$$S = -\frac{k}{4\pi} \int_M \mathrm{Tr} \left( \tilde{A} \frac{\partial \tilde{A}}{\partial t} dt \right) + \int dt \, \mathrm{Tr} \, \lambda g^{-1}(t) \left(\partial_0 + A_0\right) g(t). \tag{3.11}$$

Correspondingly, the constraint equation with a single Wilson line piercing the point $P \in \Sigma$ now becomes

$$\frac{k}{2\pi} \tilde{F} + g(t) \lambda g^{-1} \delta^{(2)}(x - P) = 0, \tag{3.12}$$

which, after fixing $P$ to the origin and using polar coordinates, has the general solution

$$\tilde{A} = \tilde{U} \tilde{d} \tilde{U}^{-1}, \tag{3.13}$$

where

$$\tilde{U} = U \exp\left( \frac{1}{k} g(t) \lambda g^{-1}(t) \phi \right). \tag{3.14}$$

$U$ is single valued on the disk. Note in particular that the holonomy of the connection around the point $P$ is given by

$$\oint A_\phi d\phi = -\frac{2\pi}{k} g(t) \lambda g^{-1}(t), \tag{3.15}$$

i.e., the holonomy is determined by the conjugacy class of the representation.

As for the disk without sources, the effective action can be derived to obtain

$$S = k S_C^+(U) + \frac{1}{2\pi} \int_{\partial M} \mathrm{Tr} \, \lambda U^{-1} \partial_t U, \tag{3.16}$$

which also depends only on the boundary values of $U$ (see appendix B for a detailed derivation). The symmetry of the effective action is slightly modified to

$$U \to \tilde{V}(\phi) U V(t), \tag{3.17}$$

except that now $V(t)$ must commute with $\lambda$. Hence, the phase space of the theory is given by $LG/T$ and the symplectic form can be derived similarly to get

$$\omega = \frac{k}{4\pi} \oint \mathrm{Tr} \left( U^{-1} \delta U \right) \frac{d}{d\phi} \left( U^{-1} \delta U \right) + \frac{1}{2\pi} \oint \mathrm{Tr} \, \lambda \left( U^{-1} \delta U \right)^2. \tag{3.18}$$

Although the additional $\lambda$ dependent term in the effective action does not make this theory the same as the CWZW model, the resulting current algebra remains the same. Hence, the Hilbert space is still a representation of the Kac-Moody algebra, except that in this case we don't obtain descendants of the identity operator, but the representation with highest weight $\lambda$, denoted by $\mathscr{H}_\lambda$[11].

By changing variables $U \to U\exp(\phi\alpha)$, where $\alpha$ is a root, we can see that the highest weight $\lambda$ is equivalent to the highest weight $\lambda + k\alpha$. Also, the theory is symmetric under the action of the Weyl group. Hence, the set of possible highest weight representations obtainable by quantizing Chern Simons theory on a disk with sources is given by

$$\lambda \in \frac{\Lambda^w}{W \ltimes k\Lambda^r}, \tag{3.19}$$

where $\Lambda^w$ and $\Lambda^r$ are the weight and root lattices respectively and $W$ is the Weyl group.

If we have multiple Wilson lines running through the bulk of $M$, the Hilbert space is the tensor product of the corresponding highest weight representations, which can be decomposed into a direct sum by using the fusion rules for the representations (alternately, through the operator algebra of the CFT dual).

### 3.3 Adding more Boundaries: The Annulus without Sources

With an additional boundary, the general solution to the constraint equation becomes

$$\begin{aligned} \tilde{A} &= \tilde{U}\tilde{d}\tilde{U}^{-1}, \\ \tilde{U} &= U\exp\left(\frac{\phi}{k}\lambda(t)\right), \end{aligned} \tag{3.20}$$

where $\lambda(t)$ can now be any arbitrary element in the Cartan subalgebra. The effective actions picks up additional contributions from the second boundary

$$S = kS_C^+(U_1) - kS_C^+(U_2) + \frac{1}{2\pi}\int \mathrm{Tr}\,\lambda(t)\left(U_1^{-1}\partial_t U_1 - U_2^{-1}\partial_t U_2\right), \tag{3.21}$$

where $U_i$'s are the values of $U$ on the two boundaries. There is a residual gauge symmetry (global symmetry from the bulk point of view) $U_i \to U_i g(t)$, $\lambda(t) \to g^{-1}(t)\lambda g(t)$. Everything is identical to the case of a disk with a source, and the boundaries result in the tensor product of the Hilbert spaces

$$\mathscr{H} = \bigoplus_{\lambda \in \Lambda^w/(W \ltimes k\Lambda^r)} \mathscr{H}_\lambda \otimes \mathscr{H}_{\lambda^*}. \tag{3.22}$$

This is the Hilbert space of the full WZW model including both left and right movers. For the annulus, the left movers appear on one boundary while the right movers appear on another.

In particular, constructing a manifold that interpolates between two annuli in the past and one in the future results in a map $\mathscr{H} \otimes \mathscr{H} \to \mathscr{H}$. Replacing the inner boundaries of the annulus with Wilson lines in representations $i, j, k$ results in maps $\mathscr{H}_i \otimes \mathscr{H}_j \to \mathscr{H}_k$ from which we can reproduce the operator algebra and the conformal blocks from Chern Simons theory[12].

## 4 Discussion

We saw how intimately connected Chern Simons theory and rational conformal field theory are by demonstrating that they are effectively different descriptions of the same underlying theory. Yet another connection between the two shows up when one quantizes Chern Simons theory on compact manifolds without boundary, in which case the Hilbert space turn out to be finite dimensional and isomorphic to the space of conformal blocks of a rational CFT. These links between Chern Simons theory and rational CFT led Moore and Seiberg to conjecture that conformal field theory is simply a generalized version of group theory[13] and all rational CFTs are classified by groups via (2+1)-dimensional Chern Simons theory[6].

Chern Simons theories, when coupled to both non-supersymmetric and supersymmetric matter, are also conjectured to have a Bose-Fermi duality[14] which has been verified in the large N limit by computing 4-point scattering amplitudes that turn out to violate crossing symmetry[15, 16]. The non-supersymmetric duality can be obtained from a supersymmetric duality between Chern-Simons theory by breaking SUSY and RG flowing[17]. Furthermore, the Bose-Fermi duality has also found relevance in condensed matter physics by tying together a web of dualities[18, 19].

## A Effective Action for $\Sigma = D$

The constraint is solved by $A_i = -\partial_i U\, U^{-1}$, for a single valued map $U : D \times R \to G$. With this substitution, we have

$$\mathrm{Tr}\, \epsilon^{ij} \left(A_i \partial_0 A_j\right) = \mathrm{Tr}\, \epsilon^{ij} \left(\partial_i U U^{-1} \partial_0 \partial_j U U^{-1}\right) - \mathrm{Tr}\, \epsilon^{ij} \left(\partial_i U U^{-1} \partial_j U U^{-1} \partial_0 U U^{-1}\right). \tag{A.1}$$

The first term in (A.1) gives a total derivative

$$\mathrm{Tr}\, \epsilon^{ij} \left(U^{-1} \partial_i U U^{-1} \partial_j \partial_0 U\right) = -\partial_j\, \mathrm{Tr}\, \epsilon^{ij} \left(\partial_i U^{-1} \partial_0 U\right), \tag{A.2}$$

since $U^{-1}\partial_\alpha U U^{-1} = -\partial_\alpha U^{-1}$. Writing the sum over indices explicitly in polar coordinates (with the convention $\epsilon^{r\phi} = +1$) gives the terms

$$+\, \partial_r\, \mathrm{Tr} \left(\partial_\phi U^{-1} \partial_0 U\right) - \partial_\phi\, \mathrm{Tr} \left(\partial_r U^{-1} \partial_0 U\right). \tag{A.3}$$

The second term vanishes under integration due to the single-valuedness of $U$. The first term gives a boundary integral

$$-\frac{k}{4\pi} \int_Y \mathrm{Tr}\, \epsilon^{ij} \left(U^{-1} \partial_i U U^{-1} \partial_j \partial_0 U\right) = \frac{k}{4\pi} \int_{\partial Y} \mathrm{Tr} \left(U^{-1} \partial_\phi U U^{-1} \partial_0 U\right) d\phi\; dt. \tag{A.4}$$

The second term in (A.1) simplifies by using the identity

$$\mathrm{Tr}\, \epsilon^{\alpha\beta\gamma} \left(U^{-1} \partial_\alpha U U^{-1} \partial_\beta U U^{-1} \partial_\gamma U\right) = 3\, \mathrm{Tr}\, \epsilon^{ij} \left(U^{-1} \partial_i U U^{-1} \partial_j U U^{-1} \partial_0 U\right). \tag{A.5}$$

Hence, we have

$$\frac{k}{4\pi} \int_Y \mathrm{Tr}\, \epsilon^{ij} \left(\partial_i U U^{-1} \partial_j U U^{-1} \partial_0 U U^{-1}\right) = \frac{k}{12\pi} \int_Y \mathrm{Tr} \left(U^{-1} dU\right)^3. \tag{A.6}$$

Adding the RHS of (A.4) and (A.6) gives the action for the CWZW model.

## B Effective Action for $\Sigma = D$ with a Source

Substituting (3.13) in (3.11), the first term in (3.11) becomes

$$\begin{aligned}\epsilon^{ij}A_i\partial_0 A_j = \epsilon^{ij}\partial_i U U^{-1}\partial_0\left(\partial_j U U^{-1}\right) \\ + \epsilon^{ij}\partial_i U U^{-1}\frac{\partial_j\phi}{k}\partial_0\left(Ug\lambda g^{-1}U^{-1}\right) \\ + \epsilon^{ij}\frac{\partial_i\phi}{k}\left(Ug\lambda g^{-1}U^{-1}\right)\partial_0\left(\partial_j U U^{-1}\right) \\ + \epsilon^{ij}\frac{\partial_i\phi}{k}\frac{\partial_j\phi}{k}\left(Ug\lambda g^{-1}U^{-1}\right)\partial_0\left(Ug\lambda g^{-1}U^{-1}\right).\end{aligned} \tag{B.1}$$

The last line of the RHS vanishes due to antisymmetry. The first line on the RHS gives the first term $kS_C^+(U)$ in the effective action (3.16). The sum of the terms on the second and third lines under the trace gives the following expression

$$\begin{aligned}\frac{2}{k}\operatorname{Tr}U^{-1}\partial_r U U^{-1}\partial_0 U g\lambda g^{-1} - \frac{1}{k}\operatorname{Tr}U^{-1}\partial_0 U U^{-1}\partial_r U g\lambda g^{-1} \\ + \frac{1}{k}\operatorname{Tr}U^{-1}\partial_r U\partial_0(g\lambda g^{-1}) - \frac{1}{k}\operatorname{Tr}g\lambda g^{-1}U^{-1}\partial_0\partial_r U.\end{aligned} \tag{B.2}$$

Adding and subtracting $(2/k)\operatorname{Tr}g\lambda g^{-1}U^{-1}\partial_0\partial_r U$ and collecting terms gives the sum of two total derivatives

$$-\frac{2}{k}\partial_r\left(\operatorname{Tr}g\lambda g^{-1}U^{-1}\partial_0 U\right) + \frac{1}{k}\partial_0\left(\operatorname{Tr}g\lambda g^{-1}U^{-1}\partial_r U\right). \tag{B.3}$$

Hence we have

$$\begin{aligned}-\frac{k}{4\pi}\int_Y \operatorname{Tr}\left(\epsilon^{ij}A_i\partial_0 A_j\right) = kS_C^+(U) + \frac{1}{2\pi}\int_Y \operatorname{Tr}\partial_r\left(g\lambda g^{-1}U^{-1}\partial_0 U\right) \\ - \frac{1}{4\pi}\int_Y \operatorname{Tr}\partial_0\left(g\lambda g^{-1}U^{-1}\partial_r U\right).\end{aligned} \tag{B.4}$$

The third term on the RHS above vanishes due to boundary conditions in $t$, giving

$$S = kS_C^+(U) + \frac{1}{2\pi}\int_{\partial Y}\operatorname{Tr}g\lambda g^{-1}U^{-1}\partial_0 U + \int dt\operatorname{Tr}\lambda g^{-1}\partial_0 g. \tag{B.5}$$

Integrating over the fields $g(t)$ gives the effective action for $U$, given by

$$S = kS_C^+(U) + \frac{1}{2\pi}\int_{\partial Y}\operatorname{Tr}\lambda U^{-1}\partial_0 U. \tag{B.6}$$

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