STRINGS, GRAVITY, AND THE WORLD WE SEE

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Abstract. This paper is a discussion of some implications of string theory on gravitational physics at low energies. We begin in Section 2 by considering closed Bosonic strings propagating in linear perturbation backgrounds as an example in which to discuss the emergence of gravity in string theory, motivate the idea of spacetime low-energy effective actions, and introduce the idea of compactification. In Section 3, we develop the theory of spinor representations in d-dimensions and apply this to worldsheet supersymmetry in $d = 2$ and spacetime supergravity in $d = 11$ and $d = 10$. In Section 4, we give a brief tour of the five $d = 10$ superstring theories and finally, in Section 5, we discuss their corresponding low energy supergravity approximations. Familiarity with quantum (conformal) field theory, general relativity, Bosonic string theory, and $d = 4$ spinor representation theory is assumed. A review of basic aspects of supersymmetry and supergravity, with a focus on $d = 4$ for familiarity, is provided in the appendices, along with a quick primer on coupling classical gravity to Fermions. The appendices are largely adapted from my final paper for PHYS 445 (albeit with some additions and edits).

Contents

1. Notation and Conventions

The convention for the metric signature is "mostly plus" ("East coast") $(-,+,+,+,\dots)$. Spacetime indices are denoted by late Greek letters (e.g. $\mu,\nu,\rho,\dots$), worldsheet Boson indices by early Roman letters (e.g. $a,b,c,\dots$), spinor and worldsheet Fermion indices by (dotted and undotted) early Greek indices (e.g. $\alpha,\beta,\gamma,\dots$), local Lorentz/orthonormal frame indices by late Roman letters (e.g $l,m,n,\dots$), Lie algebra and/or supersymmetry generator indices by middle Roman letters (e.g. $i,j,k,\dots$), and superspace indices by *capital* early Roman letters (e.g. $A,B,C,\dots$). In Minkowski space, world and tangent indices are the same and we simply use late greek indices. The worldsheet metric is denoted by h_{ab}, the spacetime metric by $\hat{g}_{\mu\nu}$ in the string frame[1] and $g_{\mu\nu}$ in the Einstein frame, and the graviton field (when considered independently from the metric) by

[1] All derived objects such as the curvature tensors or related objects such as the spacetime stress-energy tensor will be similarly "hatted".

$G_{\mu\nu}$. The spacetime dimension is denoted d. Unless otherwise noted, worldsheets are taken to be Euclidean (via Wick rotation) and target spaces taken to be Lorentzian.

The Riemann curvature tensor is given by

$$R_{\mu\nu}{}^{\rho}{}_{\sigma} = \partial_\mu \Gamma_\nu{}^{\rho}{}_{\sigma} - \partial_\nu \Gamma_\mu{}^{\rho}{}_{\sigma} + \Gamma_\mu{}^{\rho}{}_{\lambda}\Gamma_\nu{}^{\lambda}{}_{\sigma} - \Gamma_\nu{}^{\rho}{}_{\lambda}\Gamma_\mu{}^{\lambda}{}_{\sigma}$$

and the Ricci tensor is defined as the contraction of its first and third indices. The corresponding worldsheet fields will be denoted $R^{(2)}{}_{ab\ d}^{\ c}$ etc.

The γ matrices will be defined by the Clifford algebra $\{\gamma^m, \gamma^n\} = 2\eta^{mn}\mathbb{1}$. To translate our gamma matrix conventions (which match e.g. Weinberg) to that of other sources, take $\gamma^\mu \to -i\gamma^\mu$. Note that now $\gamma^\mu = \gamma^m (e_m)^\mu$ where e_m is a vielbein and m a local Lorentz index, both defined in Appendix C, and that the gamma matrices are taken to be *covariantly* constant $\nabla_\mu \gamma^\mu = 0$. Overbars will denote Dirac adjoints in the in the *body* of this paper, but right handed two-component spinors in the *appendices* of this paper.

2. Einstein and the String

2.1. **Will the Real Graviton Please Stand Up?** Consider a (closed, Bosonic) Polyakov string with worldsheet (Σ, h_{ab}) (parameterized by local coordinates σ^a) propagating in an arbitrary (d-dimensional) spacetime $(M, g_{\mu\nu})$ which is described by embedding $X^\mu : \Sigma \to M$ and (Euclidean) action

$$S_\sigma = \frac{1}{4\pi\alpha'} \int_\Sigma d^2\sigma \, \sqrt{h}\, h^{ab}\, \partial_a X^\mu(\sigma) \partial_b X^\nu(\sigma) g_{\mu\nu}(X(\sigma)) \tag{1}$$

Actions of the form (1) are called **non-linear sigma models** (hence the name S_σ; note that this has nothing to do with the worldsheet coordinates!). In this context, M is called the *target space* and Σ the *parameter space*.

Two questions immediately arise—in Minkowski spacetime, the mass spectrum of the (closed) string contained a symmetric, traceless two-tensor $G_{\mu\nu}$, which we interpreted as the gravition—the quantum of the gravitational field, which is classically represented by the metric. First off, is the graviton still contained in the (curved space) string spectrum[2]? If so, for the sake of consistency, it must be possible view the classical metric $g_{\mu\nu}$ as somehow being being composed of these quanta; otherwise our interpretation is toast and we've lost all claim to a quantum theory of gravity. Can we find such a decomposition?

Let's attempt to the answer the second question while simultaneously circumventing the first. To this end, let's return to good old flat spacetime $(\mathbb{R}^{25,1}, \eta_{\mu\nu})$ and consider a closed string propagating in this background with a plane wave graviton excitation of momentum q

$$|\, \xi; q \,\rangle = \xi_{\mu\nu} \left(\tilde{\alpha}^\mu_{-1} \alpha^\nu_{-1} + \tilde{\alpha}^\nu_{-1} \alpha^\mu_{-1} \right) |\, 0; q \,\rangle\,, \qquad \xi_{\mu\nu} = \xi_{(\mu\nu)}, \quad \eta^{\mu\nu}\xi_{(\mu\nu)} = 0$$

Note in particular that we have defined the graviton to be *Minkowsi traceless*. This will come back to haunt us soon enough.

This state corresponds to a worldsheet vertex operator (Polchinski equation (3.6.14))

$$V_{\xi,p} = \frac{g_c}{\alpha'} \int d^2\sigma \sqrt{h}\, h^{ab} \xi_{\mu\nu} \left[\partial_a X^\mu \partial_b X^\nu \, e^{ip\cdot X} \right]_r$$

[2]Determining the spectrum of a string propagating in a given curved spacetime actually turns out to be a quite nontrivial task. Things are usually the same, however, up through the first level, and so we should still expect to see a graviton.

where $[\,\cdot\,]_r$ denotes renormalization under some regularization scheme (we can always exploit Weyl invariance to make the worldsheet locally flat and thus recover the standard expression from the state-operator correspondence with renormalization given by normal ordering) and g_c is the *closed string coupling constant* which is defined based off the normalization of the vertex operator for the tachyon. But this looks wildly familiar! Sending[3] $\xi \to \frac{1}{4\pi g_c}\xi$ we find that exponentiating the plane wave graviton insertion (i.e. inserting a coherent graviton state) in the S-matrix simply gives us a perturbation of the background metric

$$\eta_{\mu\nu} \to \eta_{\mu\nu} - \xi_{\mu\nu} e^{ip\cdot X}$$

2.1.1. *A Question.* Reversing the argument, for any perturbation $\delta g_{\mu\nu}$ of the background metric, the path integral goes as

$$\exp\left[-S_\sigma\right] = \exp\left[-S_{\text{Poly}}\right] \exp\left[-\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{h}\, h^{ab}\delta g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\right]$$

If δg is Minkowski traceless[4], we can Fourier decompose δg into a superposition of plane wave graviton vertex operators and thus the right-multiplying term will again describe a coherent state of (a superposition of) plane wave gravitons.[5].

Thus we see that (closed, Bosonic, first quantized) String theory is *at least* a theory of *linearized* quantum gravity. In this context, we actually have a pretty nifty interpretation—the string propagates in a background which is described by its own massless fluctuations. This is also satisfying since the very idea of the graviton first arose in—and, classically only makes sense in—the context of linearized gravity. It is already well known how to quantize these "linear-perturbation gravitons" in the naïve field-theoretic way, but seeing them arise so naturally here suggests that we might be on the right track. Furthermore, one of the main reasons quantum gravity is so hard in the first place is that the naïve graviton theory is hopelessly nonrenormalizable. String theory approaches the problem more subtly, and thus offers the possibility of UV-finiteness (which makes sense, at least heuristically, since the string length provides a natural ultraviolet cutoff).

2.1.2. *More Than Meets the Eye?* If it is *not* possible to always make δg appropriately traceless, a new question arises—how do we go beyond linearized gravity? One interpretation, suggested by Polchinski[6][14], is that we already have all the ingredients that we need—the metric just descends from *both* the dilaton and the graviton, working in tandem to give us a whole symmetric two-tensor. In this scenario, the dilaton includes both the diagonal part of $\delta g_{\mu\nu}$ and the usual "dilaton field" Φ (see below). There is also another possible interpretation[7], which comes from a generalization of our string theory called *string field theory*. In our "first quantized" string theory, we only have (manifestly) the one string we put in by hand (ignore for now the vacuum excitations that contribute to scattering). Perhaps it is simply the case that this poor lonely string can't build an entire metric on its own. A string field theory, on the other hand, offers a "second quantized" formalism, in which we can create and destroy entire strings, and there is hope that we can now use an arbitrarily large number of strings to build up our infinitesimal variations into genuine, general, Einstein metrics. In particular, within this formalism, BRST quantization (which projects out the nonpropagating

[3] This is the normalization we should have used in the first place, we just didnt know it yet!

[4] As we will see, in string theory, Ricci flat metrics will actually *all* comprise valid expansion points in the space of metrics. Thus it is more appropriate to expand about the nearest Ricci flat background (which will not always be Minkowski) and require tracelessness with respect to *that* background metric.

[5] I only know of this happening in general for linearized gravity (where we can always go to transverse traceless gauge); if this *is* possible in general, then it would immediately show string theory to be a full theory of *nonlinear* (rather than linearized) quantum gravity.

[6] Page 109, in reference to equation (3.6.22c)

[7] This idea mainly came from Christian Ferko

degrees of freedom in our states) projects out the trace of the metric[8]. We thus can at least return to the idea that the graviton alone describes the metric—but only its physical, propagating degrees of freedom.

2.1.3. *Room for Progress.* If such a decomposition *is* possible, one might expect that string theory should be (in some sense) background independent (e.g. it seems reasonable that string theory should be independent of *Ricci flat* background). While it is perfectly fine to prescribe some initial classical spacetime background for our string(s) to propagate on, the actual metric configuration, taking into account the back-reaction of the string(s) *onto* this spacetime, should be allowed to dynamically vary (especially in string field theory where we might have arbitrarily many such strings). A background independent formulation of string theory would be very helpful in describing this phenomena, in addition to being highly desirable on more general grounds. This issue is intimately tied to the fact that, as we will see, a non-Minkowski spacetime background corresponds to an *interacting* field theory on the worldsheet, and, in particular, a non Ricci-flat spacetime background to an interacting, *nonconformal* field theory on the worldsheet. On the worldsheet, the main way we know how to tackle this issue is to restrict our attention to worldsheet field theories that are small deviations from a given conformal field theory (so that we can e.g. use its BRST operator in constructing the physical Hilbert space). But this precisely the same as picking a fixed background Ricci flat metric and considering small perturbations about it. These issues, in the context of string field theory, are expanded upon in the excellent review by Zwiebach [21].

2.2. **Averting Disaster: Conformal Invariance and the β-Function.** Since the action for the nonlinear sigma model (1) action is no longer quadratic[9] in X^μ, it describes an *interacting* worldsheet field theory. To understand what's going on, let us decompose our fields into classical, solitonic solutions $x_0^\mu(\sigma)$ and quantum fluctuations $\sqrt{\alpha'}Y^\mu(\sigma)$ (the constant out front makes our fluctuations dimensionless so we can talk about their "size"). For our own ease we take x_0^μ to be the constant solution. We can now Taylor expand the "interaction" with the metric to return to the more familiar realm of constant polynomial couplings (but at the price that we will have infinitely many of them)

$$g_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu = \alpha' \left(g_{\mu\nu}(x_0) + \sqrt{\alpha'}\,\partial_\rho g_{\mu\nu}(x_0) Y^\rho(\sigma) + \frac{\alpha'}{2}\partial_\tau \partial_\rho Y^\rho(\sigma) Y^\tau(\sigma) + \dots \right) \partial_a Y^\mu \partial_b Y^\nu$$

The (effective, dimensionless) coupling constant will be of order $\sqrt{\alpha'}/R_c$ where R_c is the typical radius of curvature of the target (this is because $\partial_\rho g_{\mu\nu} \sim R_c^{-1}$). Thus the worldsheet theory will be weakly coupled precisely when the string is small compared to the radius of curvature, or, equivalently, if the metric does not appreciably vary over the length of the string. At small coupling, we can use perturbation theory on the worldsheet, but, more importantly, we can ignore the internal structure of the string, which now simply provides a natural ultraviolet cutoff. We can thus use *low energy effective field theory.* Note that we have implicitly used the fact that, at weak coupling, no massive string states are created, and so it is consistent to restrict our attention to massless backgrounds only[10]. At large coupling, we get a notion of *"stringy geometry"*, since the string is able to probe the ambient geometry with potentially high resolution (if the string length is small); the point-particle, by comparison, cannot probe the ambient geometry *at all* since it is *too* small.

Note that string perturbation theory is now a *double expansion* (like the λ-J expansion in QFT) in both the worldsheet coupling $\frac{\sqrt{\alpha'}}{R_c}$ and the string coupling $g_s = e^\Phi$. The former counts loops in the CFT and the latter loops in the topological worldsheet expansion. It is important to note

[8] See page 53 of [16]

[9] Hence *nonlinear* sigma model

[10] As we will soon do; we are currently being even more restrictive and considering only *gravitational* backgrounds.

that, since all interactions have dimension two, the nonlinear sigma model above (and its further generalization which we will cover in the next section) is a *renormalizable* theory.

We've sure made a lot of fuss this quarter about the critical dimension, $d = 26$. We finally saw that this ended up really just being an manifestation of the need for a worldsheet CFT with $c_{\text{tot}} = 0$ in order to cancel the Weyl anomaly $\langle T^a{}_a \rangle = -\frac{c}{12} R^{(2)}$. Note that it is crucial here that the worldsheet *quantum* field theory be, in particular, a worldsheet *conformal* field theory. Since everything we did was with the Polyakov action, it is highly likely that something went terribly wrong in the transition to the sigma model[11]. In fact, we just saw in the previous section that the sigma model action violently changes the quantum field theory that lives on our worldsheet. What has happened and how do we fix it?

Let's first review an important fact. For a general QFT, conformal invariance implies scale invariance, but not the other way around. We are wildly fortunate, however, that for a *two dimensional, unitary QFT with compact spatial dimensions*[12], scale invariance implies conformal invariance. But scale invariance—that our theory looks the same at all scales—simply means that there is no RG flow. The β-function needs to vanish. We have found our problem, and the Weyl anomaly generalizes to

$$\beta_{\nu\rho}(G;\mu) = \mu \frac{\partial}{\partial \mu} G_{\nu\rho}(X;\mu) \tag{2}$$

(where we have noted that there really is just one, albeit nonconstant, coupling—G). In order to avert disaster, we must ensure that this vanishes.

2.2.1. *Computing the One Loop β-Function.* Assume for now that there is no background B or Φ fields, and let X^μ be, in particular, Riemann normal coordinates about x^μ so that

$$g_{\mu\nu}(X) = \eta_{\mu\nu} - \frac{\alpha'}{3} R_{\mu\lambda\rho\nu}(x) Y^\lambda Y^\rho$$

To quartic order in the fluctuations, we get that

$$S \supset \frac{1}{4\pi} \int d^2\sigma \sqrt{h}\, h^{ab} \left(\eta_{\mu\nu} - \frac{\alpha'}{3} R_{\mu\lambda\rho\nu}(x) Y^\lambda Y^\rho \right) \partial_a Y^\mu \partial_b Y^\nu$$

which, in terms of the worldsheet CFT, gives us a four point vertex

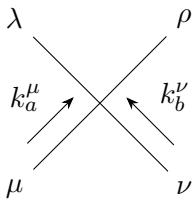
[11] There might be the idea that, in the transition from the Polyakov action to the sigma model, we no longer require Weyl invariance. This is simply not so, which we can see by simply going back to the actual reason why we needed Weyl invariance in the first place—the actual action describing the string is the Nambu-Goto action

$$S_{\text{NG}} = \frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{X^* g}$$

(X^* the pullback of the embedding) which is, among other things, generally covariant. The Polyakov action was obtained by introducing the auxiliary worldsheet metric h_{ab} which allowed us to pull our matter fields, X, out from under the square root, but this was done *in flat spacetime*. Using this trick— introducing an auxiliary metric to pull our matter fields out of the square root—is completely general, and, in a general spacetime, gives us precisely our original σ model action (without B and Φ fields). This trick will only gives us the same physics as the NG action if the auxiliary degrees of freedom are fixed and only serve to act as Lagrange multipliers. But this is precisely equivalent to demanding Weyl invariance of the new action.

[12] c.f. Polchinski's seminal paper "Scale and Conformal Invariance in Quantum Field Theory" [13]

(3)



$$\sim \alpha' h^{ab} k_a^\mu k_b^\nu R_{\mu\lambda\rho\nu}$$

where k_a^μ is the worldsheet 2-momentum of the coordinate field X^μ.

We can now calculate the one loop correction to the propagator

$$\left\langle Y^\lambda(\sigma) Y^\rho(\sigma') \right\rangle = -\frac{1}{2}\eta^{\lambda\rho} \ln\left|\sigma - \sigma'\right|^2$$

which contains

$$\alpha' h^{ab} k_a^\mu k_b^\nu R_{\mu\lambda\rho\nu} \lim_{\sigma'\to\sigma} \left\langle Y^\lambda(\sigma) Y^\rho(\sigma') \right\rangle = \alpha' h^{ab} k_a^\mu k_b^\nu R_{\mu\lambda\rho\nu} \lim_{\Lambda\to\infty} \int^\Lambda \frac{d^2p}{(2\pi)^2} \left\langle Y^\lambda(p) Y^\rho(p) \right\rangle$$

To regulate this explicitly, we work with dimensional regularization

$$\lim_{\sigma'\to\sigma} \left\langle Y^\lambda(\sigma) Y^\rho(\sigma') \right\rangle = 2\pi\eta^{\lambda\rho} \lim_{\epsilon\to 0} \int \frac{d^{2+\epsilon}k}{(2\pi)^{2+\epsilon}} \frac{e^{ik\cdot(\sigma-\sigma')}}{k^2} \underset{\sigma\to\sigma'}{\longrightarrow} \lim_{\epsilon\to 0} \frac{\eta^{\lambda\rho}}{\epsilon}$$

and to cancel this divergence, we split the metric into a physical part and a counterterm part such that

$$\delta R_{\mu\lambda\rho\nu}(x) Y^\lambda Y^\rho \partial_a Y^\mu \partial_b Y^\nu = -\frac{1}{\epsilon} R_{\mu\nu} \partial_a Y^\mu \partial_b Y^\nu$$

which can be accomplished by wavefunction renormalization $Y^\mu = Y_P^\mu + \frac{\alpha'}{6\epsilon} R^\mu{}_\nu Y^\nu$ and coupling renormalization $G_{\mu\nu} = (G_P)_{\mu\nu} + \frac{\alpha'}{\epsilon} R_{\mu\nu}$.

In dimensional regularization, the β-function is given by the coefficient of the purely divergent part of the coupling (this will be important in the next section), i.e.

$$\beta_{\mu\nu}(G) = \alpha' R_{\mu\nu}$$

2.3. **A Surprise.** Look at what we just found. Cancelling the Weyl anomaly[13] (at one loop) is completely equivalent to the target space being Ricci flat. What's more, we are working in a vacuum target (there are no fields present other than $g_{\mu\nu}$!), and so we that, in this case, cancelling the Weyl anomaly at one loop is equivalent to *satisfying Einstein's equation*

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{T}{d-2} g_{\mu\nu} \right) = 0$$

2.3.1. *Correcting Mr. Einstein.* Since we are looking at a "quantum theory of gravity", it makes sense to look for "quantum corrections" to the Einstein equations. This simply comes from looking at the β-function above at higher orders in the expansion parameter $\frac{\sqrt{\alpha'}}{R_c}$. For example, at two-loop order we have

$$\beta_{\mu\nu} = \alpha' R_{\mu\nu} + \frac{\alpha'^2}{2} R_{\mu\lambda\rho\sigma} R_\nu{}^{\lambda\rho\sigma} + \cdots = 0$$

This makes sense, since these corrections will only be appreciable when the variation of curvature becomes comparable to the string length; for appropriately small string lengths (e.g. near [14] Plank scale), this is exactly where we expect quantum corrections to GR to kick in. Such corrections to

[13] In the absence of background B and Φ fields, whose effects are expected to be small, at typical energy levels and circumstances, in the classical limit

[14] But not at! In fact, much less "near" than we would expect—see below.

the Einstein equations also appear for the heterotic string starting at two loops and for the type II string starting at four loops.

2.4. **Strings in Background Fields.** Given the interpretation above, for consistency's sake it seems appropriate to consider the string to be propagating in a background[15] described by *all* its massless excitations[16]

$$S_\sigma = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\,\sqrt{h}\,\left\{\left(h^{ab}\hat{g}_{\mu\nu}(X) + i\bar{\epsilon}^{ab}B_{\mu\nu}(X)\right)\partial_a X^\mu \partial_b X^\nu + \alpha' R^{(2)}\Phi(X)\right\}$$

As we saw in class, the Dilaton $\Phi(X)$ determines the string coupling constant $g_s = e^{\langle\Phi\rangle}$[17]. Note that this constant is determined by the VEV of a dynamical field, again suggesting the idea that string theory is free of any (continuous) free parameters other than the string length $\ell_s = \sqrt{\alpha'}$. Note that this means that (topological) perturbation theory is only valid if the string is confined to regions of spacetime where $e^{\Phi(X)} \ll 1$.

The action (4) is both symmetric under spacetime diffeomorphisms (general covariance) and under the gauge transformation $B \to B + d\Lambda$ ($\Lambda \in \Omega^2(\Sigma)$) of the Kalb-Raymond field, under which the string can carry charge. The Kalb-Raymond Field and interaction is the first of many generalizations of the standard one-form gauge symmetry (such as, e.g the photon A_μ) to higher dimensional p-form gauge symmetries[18]. Is interesting to note that the field strength $H = dB$ of the Kalb-Raymond field, when considered in tandem with the graviton/metric, essentially acts like a torsion term augmenting (after raising an index) the spacetime Levi-Civita connection. This allows for torsionful extensions of general relativity (such as full Einstein-Cartan theory) where now the torsion can be tuned to be small from stringy considerations. For this reason, H is sometimes called the *torsion form*.

2.5. **The Physics We See: The Low-Energy Effective Action.** The Dilaton breaks manifest Weyl invariance *at order* α' (notice the prefactor of α' in the Dilaton term in the action), which must be compensated by quantum (one loop, in the α' expansion on the worldsheet) corrections to g and B. For the special case where δg, B and Φ are "small", i.e.

$$\hat{g}_{\mu\nu}(X) = \eta_{\mu\nu} - 4\pi g_c\,\xi_{(\mu\nu)}e^{ik\cdot X}, \quad B_{\mu\nu}(X) = -4\pi g_c\,\xi_{[\mu\nu]}e^{ik\cdot X}, \quad \text{and} \quad \Phi(X) = -4\pi g_c\Phi_0 e^{ik\cdot X}$$

we find a Weyl anomaly of the form

$$T^a{}_a = -\frac{1}{2\alpha'}\,\beta^G_{\mu\nu}\,h^{ab}\partial_a X^\mu\partial_b X^\nu - \frac{i}{2\alpha'}\,\beta^B_{\mu\nu}\,\bar{\epsilon}^{ab}\partial_a X^\mu\partial_b X^\nu - \frac{1}{2}\,\beta^\Phi R^{(2)}$$

[15]The classical spacetime metric, g, differs from the metric $\hat{g}$ in the sigma model for reasons to be discussed soon.

[16]The $\bar{\epsilon}$ here carries the unorthodox normalization $\sqrt{h}\,\bar{\epsilon}^{12} = 1$, which stems from the fact that the Kalb-Raymond interaction should naturally be written as the simple integral over the worldsheet of the Kalb-Raymond two form

$$S_{\text{KR}} = \int_{X(\Sigma)}\mathbf{B} = \int_{X(\Sigma)} B_{\mu\nu}\,dX^\mu\wedge dX^\nu = \int_\Sigma X^*\mathbf{B} = \int_\Sigma B_{\mu\nu}\,\partial_a X^\mu\partial_b X^\nu d\sigma^a\wedge d\sigma^b \tag{4}$$

The factor of $\bar{\epsilon}$ is required to "factor out" the wedge product, while the accompanying factor of i is just a remnant of our Wick rotation, and will return to unity if we go back to Lorentzian signature.

[17]More generally, we have $g_s = e^{\Phi_0}$ where Φ_0 is the asymptotic value of the Φ field, when defined.

[18]Just as a one-form gauge field is the connection on some principal bundle, the Kalb-Raymond field is a connection on a mathematical object called a *gerbe*.

where the coefficients $\beta^{(\ )}$[19] (see the previous section) provide us with the β-function(al)s of the theory, which, to $\mathcal{O}(\alpha', \tilde{\nabla}^2)$[20] include

$$\beta^G_{\mu\nu} = \alpha' \hat{R}_{\mu\nu} + 2\alpha' \hat{\nabla}_\mu \hat{\nabla}_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\rho} \hat{H}_\nu{}^{\lambda\rho} + \dots$$

Canceling the Weyl anomaly now gives us Einsteins equation for (Kalb-Raymond and dilaton) matter with

$$8\pi G \left(\hat{T}_{\mu\nu} - \frac{\hat{T}}{d-2} \hat{g}_{\mu\nu} \right) = \frac{1}{4} H_{\mu\lambda\rho} \hat{H}_\nu{}^{\lambda\rho} - 2\hat{\nabla}_\mu \hat{\nabla}_\nu \Phi$$

It is interesting to note that the equation $\beta^B_{\mu\nu} = 0$ can be viewed of the generalization of the spacetime Maxwell equations for the three-form Kalb-Raymond field. It is important to note that, because of the necessity of canceling the Weyl anomaly, strings can *only* consistently propagate in a background which satisfies the appropriate equations of motion. This is part of the reason why off-shell vertex operators don't quite make sense in string theory.

More generally, we can view the equations $\beta^{(\ )} = 0$ as coupled equations of motion for the fields g, B, and Φ. A spacetime physicist, who knows nothing about the string or the worldsheet[21] and who only has access to our current experimental technology will only see a *spacetime*, *low-energy*, *effective* action, or **low-energy-effective action** (to draw contact with standard QFT and save a few characters), which should give rise to the same equations of motion as above. An example is

$$S_{\text{Eff}} = \frac{1}{2\kappa_0^2} \int d^d X \sqrt{-\hat{g}}\, e^{-2\Phi} \left(\frac{2(26-d)}{3\alpha'} + \hat{R} - \frac{1}{12} H_{\mu\nu\lambda} \hat{H}^{\mu\nu\lambda} + 4\, \hat{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \mathcal{O}(\alpha') \right) \tag{5}$$

where κ_0 is some normalization. On dimensional grounds we expect that $\kappa_0^2 \sim \ell_s^{d-2} \sim \alpha^{\frac{d-2}{2}}$. At low energies, scattering predictions made by S_{Eff} agree with those made by the string S-matrix. We note for now that the low energy effective action for the superstring has a contribution of nearly identical form, except with a first piece depending on $d-10$ rather than $d-26$.

2.5.1. *The Einstein Frame.* This whole time, we have been writing the metric with a funny hat. What gives? Well, there's something equally funny going on—the metric term in the action does not take standard Einstein-Hilbert form, and the dilaton kinetic term is of the wrong sign. Something seriously weird seems to be going on.

First let's think about *why* the action is the way it is. The factor of $e^{-2\Phi}$ out front comes from the fact that the action has been computed at tree level in string perturbation theory. The constant mode of the dilaton simply provides the *constant* part of the string coupling and is innocuous. It is thus helpful to isolate the varying part of the Dilaton via the field redefinition $\varphi = \Phi - \langle \Phi \rangle$. We can similarly redefine[22]

$$g_{\mu\nu} = e^{-\frac{4\varphi}{d-2}} \hat{g}_{\mu\nu} \tag{6}$$

[19] We absorb the flat spacetime anomaly, including contributions from the ghost fields, into the β function for Φ.

[20] We have also ignored terms of higher than second order in derivatives, since they will come in at higher than linear order in the worldsheet α'/R_c expansion

[21] Dr. Sethi likes to mention this poor guy a lot.

[22] Note that this is a conformal transformation $g = \tilde{\Omega}^2 \tilde{g}$ and thus does not solve the trace problem,

$$\tilde{\eta}^{\mu\nu} \tilde{g}_{\mu\nu} \neq 0 \implies \eta^{\mu\nu} g_{\mu\nu} = \tilde{\Omega}^2 \eta^{\mu\nu} \tilde{g}_{\mu\nu} = \tilde{\Omega}^4 \tilde{\eta}^{\mu\nu} \tilde{g}_{\mu\nu} \neq 0$$

Note that this is just a conformal rescaling with conformal factor $\Omega = e^{2\omega}$ with $\omega = -\frac{2\varphi}{d-2}$. We thus get that

$$R = e^{\frac{4\varphi}{d-2}} \left(\hat{R} + 4\,\frac{d-1}{d-2}\hat{\nabla}_\mu \hat{\nabla}^\mu \varphi \, - 4\,\frac{d-1}{d-2}\partial_\mu \varphi\, \partial^\mu \varphi \right)$$

and so

$$(7) \quad S_{\text{Eff}} = \frac{1}{2\kappa^2}\int d^d X \sqrt{-g}\left[\frac{2(26-d)}{3\alpha'} e^{\frac{4\varphi}{d-2}} + R - \frac{1}{12} e^{-\frac{8\varphi}{d-2}} H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{4}{d-2}\partial_\mu \varphi\, \partial^\mu \varphi \, + \mathcal{O}(\alpha') \right]$$

where now *all indices have been raised and lowered with $g^{\mu\nu}$ and $g_{\mu\nu}$*. The action for $g_{\mu\nu}$ is the Einstein-Hilbert action; $g_{\mu\nu}$ is the metric that we see in general relativity. In this context, the choice of field definitions is usually referred to as a choice of *frame*. This frame is known as the **Einstein frame** and the former as the **string frame**. $\hat{g}_{\mu\nu}$ is called the string metric or the sigma model metric and $g_{\mu\nu}$ the Einstein metric.

The gravitational coupling is given by $\kappa^2 = \kappa_0^2\, e^{2\langle \Phi \rangle} \sim \ell_s^{d-2} g_s^2$. In d dimensions, we usually identify

$$\kappa^2 = 8\pi G_{N,d}$$

where $G_{N,d}$ is the d-dimensional Newton constant. If we used this to define the d-dimensional Planck scale, we see that the perturbative regime is actually the regime where the string is still large compared to the d-Planck scale (so that the notion of geometry, and hence of a continuous string, still makes sense)

$$g_s \ll 1 \implies \ell_{p,d} \ll \ell_s$$

Thus, actually studying the bona fide *quantum* aspects of gravity requires the non-perturbative regime of string theory.

2.5.2. *A Victory and a Loss?* Note that the reason we were able to consider two different metrics—$\hat{g}$ and g—came down to the fact that we had a background massless scalar field Φ. In some sense, we can use Φ as a "ruler" of sorts to compare metrics. In another, more definite sense, the long range attractive forces mediated by Φ can "mix" with the gravitational force and violate the equivalence principle (see the next section for an example). In order to protect the equivalence principle at the scales where it has been supported, we must find a way to make Φ sufficiently massive, which has been found in superstring theory.

2.6. **The Dilaton and Violations of the Equivalence Principle.** Let's explore in a bit more detail what was meant by the comment that a massless dilaton violates the (weak) equivalence principle.

In particular, a massless dilaton violates the universality of freefall. To see this in action, let's consider a simplified (Einstein frame, compactified) effective action of the form

$$S_{\text{Eff, toy}} = \frac{1}{8\pi G_N}\int d^4 x \sqrt{-g}\left(R - 2\nabla_\mu \varphi\, \nabla^\mu \varphi \, - \bar{\Psi} D \Psi - \frac{1}{4g_0^2}\,\Xi(\varphi\,) F^2 \right)$$

where Ψ represents all matter fields, D is the appropriate representations of the (spacetime + gauge) covariant derivative, and F the gauge field strength (e.g. for $SU(3)\times SU(2)\times U(1)$). Ξ is proportional to the Weyl transformation taking us from the string frame to the Einstein frame, $g_{\mu\nu} \propto \Xi(\Phi)\hat{g}_{\mu\nu}$ and $\Xi(\varphi\,) \coloneqq \Xi[\Phi(\varphi\,)]$. Thus the gauge field couples to the dilaton in such a way that the effective gauge coupling is given by $g^{-2} = g_0^{-2}\,\Xi(\varphi\,)$, which, in the case of the standard model, would imply that the QCD mass scale goes as

$$\Lambda_{\text{QCD}} \sim \frac{C}{\sqrt{\Xi(\varphi\,)}}$$

where C is some constant depending on the RNG flow behavior of the QCD coupling constant and the choice of string unification scale. Under the assumption of a universal $\Xi(\varphi)$, the masses of all particles will depend on φ only though Ξ, i.e. $m(\varphi) = m[\Xi(\varphi)]$.

Now consider two (nearly particulate) masses (whose masses primarily come from quarks; for definiteness consider nonrelativistic Hadrons), A and B coupled by interaction potential (we work, for simplicity, in the Newtonian regime)

$$V_{AB} = -\frac{G_{AB} m_A m_B}{r_{AB}}$$

where G_{AB} is the Newton "constant", now allowed to vary from interaction to interaction, which we guess to parameterize as

$$G_{AB} = G_N(1 + \alpha_A \alpha_B)$$

Consider now adding an external mass m_E. Masses A and B will fall in the gravitational field of m_E with accelerations a_A and a_B differing by[23]

$$\left(\frac{\Delta a}{a}\right)_{AB} := 2\,\frac{a_A - a_B}{a_A + a_B} \simeq (\alpha_A - \alpha_B)\alpha_E$$

The massless dilaton will invoke an equivalence principle violation of the form

$$\left(\frac{\Delta a}{a}\right)_{AB} \propto [k(\varphi - \varphi^*)]^2$$

where φ is the present value of the dilaton, φ^* a point which maximizes Ξ[24], and k the curvature of $\ln \Xi$ at φ^*. Note that, in this model, a constant dilaton fixed at $\varphi = \varphi^*$ would not affect the equivalence principle, which would then remain "non-anomalous" at the quantum level. For more details, see [7].

2.7. **The World We See: Compactification and Four Dimensional Physics.** The Bosonic string lives in $25 + 1$ dimensions and the superstring in $9 + 1$. Every observation we've ever made does not suggest that we live in something other than $3 + 1$ (i.e. we observe three macroscopic spacelike dimensions and one macroscopic timelike dimension). If string theory is really our theory of everything, there must be some way to reconcile these facts. How do we do this?

Since gravity is dynamical in string theory, it is possible for the "extra dimensions" to curl up and/or "compactify". In some situations this is kinematically prescribed and in others dynamically required. Let's looks for some toy model examples of the former for the vacuum effective action.

When the effect of matter is small, at lowest order in α', the EOM for gravity is simply the requirement of Ricci flatness

$$R_{\mu\nu} = 0$$

We seek a solution geometry of the form $(\mathbb{R}^{3,1} \times M_I, g^{(3,1)} \oplus g^{(22)})$ where M_I is a compact, 22 dimensional manifold equipped with Ricci flat metric $g^{(22)}$. The simplest such manifold is just the 22-dimensional torus, $M_I = T^{22}$, equipped with a flat metric (this provides a so-called *toroidal compactification*). In general, we find use in considering compactifications involving the so-called **Calabi-Yau** manifolds—compact, *complex* manifolds with vanishing first Chern class[25]. The Standard Model has been well tested to energies of around 1-10 TeV (the most recent run, which began in 2015, has been focusing on the 13 TeV range), so that, naïvely, if the standard model particles

[23] Note that, unless $\alpha_A \neq \alpha_B$ for *some* pair of objects, G_{AB} will just be a universal constant.

[24] In many gravity-dilaton-matter cosmological models, the dilaton is dynamically driven towards maxima of Ξ.

[25] The vanishing of the Chern class, in particular, guarantees the existence of Ricci flat metrics by *Yau's theorem*

can access X, the maximum length scale must be $\lesssim (\text{TeV})^{-1} \sim 10^{-16}$cm, though much research has also been done on scenarios that allow for "large" extra dimensions as well.

There is also the potential that our world is some four-dimensional submanifold of an ambient d-dimensional universe; these are the so-called "brane-world" models. These are interesting since they allow for the possibility that, while most of these physics we see is restricted to some D_4-brane, *gravity*, at high-energies, might be allowed to "leak" into the "bulk" spacetime, allowing for entirely new, potentially testable, gravitational physics; see e.g. [10]. In these scenarios, the constraints on the extra dimensions are much weaker, since the main input must come from gravitational experiments, which simply cannot be done to the same precision as particle experiments with our current knowledge. Because of this, the bound on the length scale in these scenarios is something more like $\lesssim 10^{-5}$cm, which leaves lots of wiggle room[26].

Extra dimensions (though often much less than 6) have been researched since the 1920's—when Kaluza-Klein theory first came into being—as a potential source of new physics. There is extensive literature on this, mainly in terms of particle phenomenology, on the ArXiv. On the gravity side, there is also research being done on the potential for observable signatures of extra dimensions in gravitational waves[1].

There are quite a few technical problems that can occur during compactification, and so one often restricts *explicit* consideration to *static* spacetimes with time dependence introduced a fortiori via the low energy effective action (which is fine so long as the time scale is much larger than the string scale, which is often the case in cosmological applications). The static constraint is mainly there to enforce that the only X^0-dependent term in the action is its kinetic term. Under this assumption, we can relax the 26 fields X^μ of our theory to be, instead X^0 and any other unitary CFT with $c = \tilde{c} = 25$ (this will ensure that we maintain Diff×Weyl invariance and controls the spacetime inner product between states). In particular, we can choose to utilize the fields X^μ for $\mu = 0, 1, 2, 3$ and then some other compact (i.e. of discrete spectrum) unitary CFT with $c = \tilde{c} = 22$. The only problem with this is that, for two dimensional CFTs, it is not clear that these really constitute *different* theories at all. In two dimensions, many—if not all—string theories with a given set of (worldsheet) gauge symmetries and worldsheet topology represent different *vacua* of the same *theory*.

2.7.1. *A Toy Model for Compactification.* Consider the Einstein-Hilbert part of the low-energy effective action in the Einstein frame (7)

$$S_{\text{EH}} = \frac{1}{2\kappa^2}\int d^{26}X\sqrt{-g}\,R = \frac{\text{vol}(M_I)}{2\kappa^2}\int d^4X\sqrt{-g^{(3,1)}}R^{(3,1)}$$

where we have "integrated out" the extra dimensions (ignoring the many possible moduli of X along the way). The effect of the extra dimensions in this model is to modify the Newton constant

$$8\pi G_{N,4} = \frac{\kappa^2}{\text{vol}(M_I)} \implies G_{N,4} = \frac{1}{\text{vol}(M_I)}\,G_{N,26}$$

which gives us a plank scale

$$\ell_{p,4} \sim \frac{g_s\ell_s^{12}}{\sqrt{\text{vol}(M_I)}}$$

subject to the constraints that the length scales of M_I be greater than ℓ_s and $g_s \ll 1$ (so that our analysis is consistent), which tells us that that $\ell_{p,4} < \ell_s$ (again, this is needed for consistency, so that we can ignore the fine details of M_I). This shows that, even though $\ell_{p,4}$ is *quite* small, ℓ_s might

[26] pun not intended

be considerably larger due to the volume of the extra dimensions, potentially bringing us closer to the realm of experimental testability.

3. Away from the World We Know: Working in $d \neq 4$

We now take a brief detour to review the generalization of supersymmetry and supergravity to higher dimensions (the appendices mainly focus on the case $d = 4$ to avoid introducing the potentially confusing subtleties associated with extra dimensions). To begin, we first must understand the generalization of the concept of spinor representations of the $SO(3,1)$ Lorentz algebra to $SO(d-1,1)$ in d dimensions.

Note once more that, *in stark contrast with the appendices*, in the body of this paper we will use overbars to denote the Dirac adjoint. For example, $\overline{Q}$ will denote the corresponding SUSY generator in the *Dirac representation*, i.e. such that $\{Q, \overline{Q}\} = -2\not{P}$.

3.1. Spinors and γ Matrices in d Dimensions.

The Dirac γ matrices in d dimensions are simply a representation of the d-dimensional Clifford algebra

$$\{\gamma^m, \gamma^n\}_{\alpha\beta} = 2\eta^{mn}\mathbb{1}_{\alpha\beta}$$

generated by d matrices γ^m. It is clear that γ^0 is skew-Hermitian and the remaining γ^i Hermitian.

3.1.1. *Even Dimensions.* Write $d = 2k+2$ for $k \in \mathbb{N}$. We can essentially find all even dimensional representations of the Clifford algebra by essentially building Fock spaces. We first change to a basis of raising and lowering operators

$$\gamma^{0\pm} \coloneqq \frac{1}{2}\left(\pm\gamma^0 + \gamma^1\right) \quad \text{and} \quad \gamma^{i\pm} \coloneqq \frac{1}{2}\left(\gamma^{2i} \pm i\gamma^{2i+1}\right) \quad i = 1, \ldots, k \tag{8}$$

$$\{\gamma^{i+}, \gamma^{j-}\} = \delta^{ij} \tag{9}$$

and find the unique "ground state" ζ, given by $\gamma^{i-}\zeta = 0$ for all i. A basis of the representation space can be built up from acting on ζ with all possible combinations of raising operators, and the matrix elements of the γ^m can be found using the commutation relations (9). Call these basis states

$$\zeta^{(s_0, s_1, \ldots, s_k)} \coloneqq (\gamma^{k+})^{s_k + \frac{1}{2}} \ldots (\gamma^{0+})^{s_0 + \frac{1}{2}}\zeta$$

where each[27] $s_i = \pm\frac{1}{2}$ and ζ corresponds to $s_i = -\frac{1}{2}$.

One of the main cases of interest will be $d = 2$, where we get the real **Majorana-Weyl** representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We can actually use these to recursively build the γ matrices in arbitrary even dimension, via

$$\gamma^{d-2} = \mathbb{1}_{2k} \otimes \sigma_3, \quad \gamma^{d-1} = \mathbb{1}_{2k} \otimes \sigma_2, \quad (\gamma^m)_{2k+2} = (\gamma^m)_{2k} \otimes \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad m = 0, \ldots, d-3$$

The corresponding $\Sigma^{mn} = \frac{1}{4}\{\gamma^m, \gamma^n\}$ satisfy the Lorentz algebra and we see that the states of the Fock space form the 2^{k+1} dimensional *Dirac spinor representation* (an irrep) of the Lorentz algebra $SO(2k+1,1)$. While it is not possible in $d = 4$, in $d = 2$ and $d = 10$ (in general in $d = 2 \bmod 8$), we can impose both the Majorana reality and the Weyl chirality condition simultaneously, giving real **Majorana-Weyl** spinors of definite chirality.

[27] Recall the Pauli exclusion principle.

The generators $\Sigma^{2a,2a+1}$ commute and can thus be simultaneously diagonalized. Define the spin operators

$$S_a = i^{\delta_{a,0}}\Sigma^{2a,2a+1} = \gamma^{a+}\gamma^{a-} - \frac{1}{2}$$

so that $\zeta^{(s_0,\ldots,s_k)}$ is a simultaneous eigenstate of each of the S_a with eigenvalue s_a.

In (even) $d \neq 4$, the generalization of γ^5 is simply called γ:

$$\gamma := i^{-k}\gamma^0\gamma^1\ldots\gamma^{d-1} = 2^{k+1}S_0S_1\ldots S_k$$
$$(\gamma)^2 = 1, \quad \{\gamma,\gamma^\mu\} = 0, \quad [\gamma,\Sigma^{mn}] = 0$$

Note that, for even d, this gives us a notion of Weyl spinor and hence shows that the Dirac representation is reducible. In even dimensons, we define the projection operators $P_\pm = \frac{1}{2}(1\pm\gamma)$ onto subspaces of definite chirality.

3.1.2. *Odd Dimensions.* We can use γ to form representations of the Clifford algebra in odd dimensions $d = 2k+3$ by adding either $\gamma^d = \gamma$ or $\gamma^d = -\gamma$ (since these commute with the Σ^{mn} of $d = 2k+2$) to the algebra for $d = 2k+2$, giving an *unique* spinor irrep of $SO(2k+2,1)$.

3.2. **SUSY on the Worldsheet: $d = 2$.** Let's touch ground with the two-dimensional worldsheet superalgebras that we will need for the superstring. The smallest spinor irrep in $d = 2$ is the Majorana-Weyl rep from above with one Hermitian component, while the general $(\tilde{\mathcal{N}},\mathcal{N})$ algebra will have $\tilde{\mathcal{N}}$ Hermitian left-moving supercharges $\tilde{Q}^i$ and $\mathcal{N}$ Hermitian right-moving supercharges Q^j, obeying

$$\{\tilde{Q}^i,\tilde{Q}^j\} = \delta^{ij}\left(P^0 - P^1\right), \qquad \{Q^i,Q^j\} = \delta^{ij}\left(P^0+P^1\right), \quad \text{and} \quad \{\tilde{Q}^i,Q^j\} = Z^{ij}$$

where Z^{ij} is a (now unconstrained) central extension term. This is the algebra satisfied by the superconformal generators on the superstring worldsheet.

3.3. **Supergravity in $d = 11$.** The largest supersymmetry algebra allowed in $d = 4$ is $\mathcal{N} = 8$ with 32 supercharges, since anything more would require the existence of particles with helicity greater than 2 (for which it is currently believed to be impossible to construct nontrivial interactions). This same limit will hold in higher dimensions, since we can always reduce to four via toroidal compactification (unless no supersymmetry survives the compactification, which is pointless). Since spinor representations for $d \geq 12$ are 64-dimensional or larger, we see that the highest dimension that we can consistently consider supersymmetry in is $d = 11$ (note that in this dimension, since all spinors are 32 dimensional, $\mathcal{N} = 8$ is also the *smallest* allowed supersymmetry, and so we see that this is the *unique* physically consistent supergravity of top dimension). This exceeds the critical superstring dimension by one, but is still a worthwhile starting point.

The Majorana supercharge satisfies the algebra $\{Q_\alpha,\overline{Q}_\beta\} = -2P_\mu\gamma^\mu_{\alpha\beta}$. The massless irreps contain $\frac{1}{2}\cdot 2^8 = 128$ Bosons and Fermions each. By calculating the spins S_1 through S_4, we see that the graviton multiplet contains two Bosonic representations of $SO(9)$: the traceless symmetric two tensor (graviton) and a three form. There is a single Fermionic vector-spinor representation. In particular, the gravitino $\Psi_{\mu\alpha}$ will be the mediator of local supersymmetry.

There is a unique action (with two or fewer derivatives) with Bosonic part

$$S_{\text{SUGRA, 11}} = \frac{1}{2\kappa^2}\int d^{11}x\sqrt{-g}\left(R - \frac{1}{2}|F_4|^2\right) - \frac{1}{12\kappa^2}\int A_3\wedge F_4\wedge F_4 \tag{10}$$

where A_3 is a three-form potential and $F_4 := dA_3$ its field strength. The last term is a gauge-invariant Chern-Simons term.

3.4. $d = 10$ **Type IIA SUGRA.** We can compactify the $d = 11$, $\mathcal{N} = 8$ supergravity above via *dimensional reduction*—compactifying on the torus and keeping only the massless fields. The $d = 11$ Majorana spinor SUSY generator becomes a set of two $d = 10$ Majorana-Weyl spinors Q^i_α, one of each chirality. The product of two spinors of opposite chirality yields and scalar, and of the same chirality a vector. Note that gamma matrix, γ^{10}, corresponding to the compactified dimension, reduces to the γ of the $d = 10$ algebra, giving central charge term proportional to the Kaluza-Klein momentum

$$\{Q^1_\alpha, \overline{Q}^2_\beta\} = -2P_{10}(P_+\gamma)_{\alpha\beta}$$

The dimensional reduction leaves a scalar dilaton, a Kaluza-Klein vector (one-form), a two-form, and a three-form. Supersymmetry determines the massless particle content completely, and we see that (essentially inevitably) the spectrum is identical to that of the type IIA superstring. We interpret this supergravity as corresponding to the type IIA superstring's low energy limit.

Let's now go through the motions of obtaining the action for the type IIA theory from dimensional reduction of $d = 11$, $\mathcal{N} = 8$ supergravity. The latter theory contains a metric[28] $g^{(11)}_{\mu\nu}$ and a three form $A^{(11)}_3$ and has an action whose Bosonic part is given by (10). Under dimensional reduction[29], the eleven dimensional metric, $g^{(11)}_{\mu\nu}$, is replaced by a new metric, $g_{\mu\nu} \coloneqq g^{(10)}_{\mu\nu}$, a scalar σ, and a gauge field A'_μ, which appear from considering only those eleven dimensional metrics which are invariant under translations in the compactified direction

$$ds^2_{(11)} = g^{(11)}_{\mu\nu} dx^\mu_{(11)} dx^\nu_{(11)} = g_{\mu\nu} dx^\mu dx^\nu + e^{2\sigma(x^\mu)} \left(dx^{10}_{(11)} + \rho_\nu(x^\mu) dx^\nu \right)^2$$

The gauge field $A^{(11)}_3$ reduces to two gauge fields A_3 and A_2 (which descends from the components of $A^{(11)}_3$ which lay along the compactified direction). In terms of the dilaton, $\Phi = 3\sigma/2$, the vielbein reduce via

$$(e^m{}_\mu)^{(11)} = \begin{pmatrix} e^{-\Phi/3} e^a{}_\mu & 0 \\ e^{2\Phi/3} A'_\mu & e^{2\Phi/3} \end{pmatrix}$$

The Bosonic part of the action thus splits into three parts

$$S_1 = \frac{1}{2\kappa^2_{10}} \int d^{10}x \sqrt{-g} \left(e^\sigma R - \frac{1}{2} e^{3\sigma} |F_2|^2 \right) \tag{11}$$

$$S_2 = -\frac{1}{4\kappa^2_{10}} \int d^{10}x \sqrt{-g} \left(e^{-\sigma} |F_3|^2 + e^\sigma |F'_4|^2 \right) \tag{12}$$

$$S_3 = -\frac{1}{4\kappa^2_{10}} \int A_2 \wedge F_4 \wedge F_4 = -\frac{1}{4\kappa^2_{10}} \int A_3 \wedge F_3 \wedge F_4 \tag{13}$$

We have compactified the theory on a circle of radius R and thus have defined $\kappa^2_{10} \coloneqq \kappa^2_{11}/2\pi R$; we have also defined $F'_4 \coloneqq dA_3 - A' \wedge F_3$, which we regard as a physical field strength which happens to satisfy a nonstandard Bianchi identity: $dF'_4 = -F_2 \wedge F_3$, whose origin stems from residual gauge freedom from the compactified dimension.

[28] From now on we will only consider metrics in the string frame, and hence will drop the "hats" since there will be no chance for confusion.

[29] See e.g. Section 8.1 of [15].

3.5. $d = 10$ **Type IIB SUGRA.** There is a second supergravity in $d = 10$, which is *not* obtained by compactifying an eleven-dimensional theory. This supergravity contains two superalgebras, given by $\{Q^i_\alpha, \overline{Q}^j_\beta\} = -2p_\mu(P_+\Gamma^\mu)_{\alpha\beta}\delta^{ij}$, of the same chirality. The graviton multiplet contains the graviton, two scalars, two two-forms, and a four-form (with self-dual field strength). This is the same as the massless content of the type IIB superstring, and we will again interpret this supergravity as that superstring's low energy limit.

4. The Superstring Theories

The key difference between the Bosonic string and the superstring is the addition of worldsheet and spacetime supersymmetry and thus of worldsheet and spacetime Fermions. There are two ways of doing this. The first, the so-called **Green-Schwarz (GS)** formalism, which makes spacetime supersymmetry manifest (which thus gives us spacetime Fermions *manifestly*), and the so-called **Ramond-Neveu-Schwarz (RNS)**, which adds Fermionic modes on the worldsheet (i.e. makes the worldsheet a two dimensional supermanifold, carrying a two dimensional superconformal field theory), which instead gives us *emergent* spacetime Fermions and supersymmetry. Note that the GW formalism also gives us worldsheet supersymmetry, but in GW *this* is emergent rather than manifest, the direct opposite of the RNS formalism.

The superstring has critical dimension $d = 10$ [30], has no tachyon, and still has the fields $G_{\mu\nu}$, $B_{\mu\nu}$, and Φ in its spectrum. In this context, we sometimes instead call $B_{\mu\nu}$ the Neveu-Schwarz two form. We also find not just spacetime Fermions, but additional massless spacetime Bosons. The form of these extra Bosons depends on the type of superstring theory we consider.

Let's be clear on what we mean by "type of superstring theory". The Bosonic string is a unique theory (in the sense that the one discrete choice we make—open or closed—turns out to not really matter). For the *superstring*, we can make a number of discrete choices which give rise to *different* perturbative superstring theories (although later developments suggest that they are all actually part of the same framework, that of **M-theory**).

The most important of these discrete options (let's work for now in the RNS formalism) is whether or not we add worldsheet Fermions in both the left-moving and right-moving sectors of the string, or if we choose the Fermions to all move in one direction (usually taken to be the right). This gives rise to two different classes of superstring theory

- **Type II** in which we have both left and right-moving worldsheet Fermions. The resulting spacetime has $d = 10, \mathcal{N} = 2$ supersymmetry (32 supercharges). These theories contain extra massless Bosonic states known as *Ramond-Ramond (RR) fields*. The action is given by

$$S_{\text{II}} = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{h}\, g_{\mu\nu}\left[h^{ab}\partial_a X^\mu \partial_b X^\nu + \frac{i}{2}\psi^\mu\gamma^a\partial_a\psi^\nu + \frac{i}{2}\left(\chi_a\gamma^b\gamma^a\psi^\mu\right)\left(\partial_b X^\mu - \frac{i}{4}\chi_b\psi^\mu\right)\right] \tag{14}$$

  with the leftmoving worldsheet supergravity (we are coupled to metric h^{ab}!) given by

$$\delta X^\mu = i\xi\psi^\mu, \quad \delta\psi^\mu = \gamma^a\left(\partial_a X^\mu - \frac{i}{2}\chi_a\psi^\mu\right)\xi, \quad \delta\bar{\psi}^\mu = 0$$

$$\delta g_{ab} = i\xi(\gamma_a\chi_b + \gamma_b\chi_a), \quad \delta\chi_a = 2\,\nabla_a\xi$$

[30] This is still out of the necessity of cancellation of the Weyl anomaly, but now we have to add in the $\beta\gamma$ ghosts and account for the central charge contributions from these and from the worldsheet Fermions. All in all we get

$$0 = c_{\text{Bose}} + c_{\text{Ferm}} + c_{bc} + c_{\beta\gamma} = (d + d/2) - 26 + 11 \implies d = 10$$

The rightmoving supersymmetry is analogous.

- **Heterotic** in which we only have right-moving worldsheet Fermions. More accurately, in the heterotic string, we exploit the independence of the left-moving and right-moving sectors to get *equivalent* descriptions of the string using *either* a right-moving Fermionic sector, *or* a left-moving Bosonic sector (the theory will end up containing both sectors). The resulting theory has $d = 10, \mathcal{N} = 1$ supersymmetry (16 supercharges). These theories do not contain Ramond-Ramond fields; instead each comes with a non-Abelian spacetime gauge field. The action for the Fermionic formulation of the theory is given (in worldsheet lightcone gauge/coordinates) by

$$S_{\text{Het}} = \frac{1}{\pi}\int d^2\sigma\, g_{\mu\nu}\left(2\,\partial_+ X^\mu \partial_- X^\nu + i\psi^\mu\partial_+\psi^\nu + i\sum_{i=1}^{32}\lambda^i\partial_-\lambda^i\right) \tag{15}$$

  where the λ^i are a collection of Lorentz singlets.

In each of these cases, there is one further discrete choice we can make

- **Type IIA:** The Ramond-Ramond fields are a one-form C_μ and a three-form $C_{\mu\nu\rho}$. We think of each of these as a gauge field. The theory is Poincaré invariant and has $\mathcal{N} = 8$ supersymmetry.
- **Type IIB:** The Ramond-Ramond gauge fields are a scalar C, a two-form $C_{\mu\nu}$, and a four-form $C_{\mu\nu\rho\sigma}$. The four-form has a self dual field strength $F_5 = *F_5$ (there are some subtleties involved with this; see the next section).
- **Heterotic** $SO(32)$**:** Comes with a spacetime $SO(32)$ Yang-Mills field.
- **Heterotic** $E_8 \times E_8$**:** Comes with a spacetime $E_8 \times E_8$ Yang-Mills field.

Note that both $SO(32)$ and $E_8 \times E_8$ contain the $SU(3) \times SU(2) \times U(1)$ gauge group of the standard model. In particular, Candelas, Horowitz, Strominger, and Witten discovered that the $E_8 \times E_8$ Heterotic string admits a "spontaneous" compactification from ten to four dimensions on a six dimensional Calabi-Yau manifold, with a resulting four dimensional theory that has gauge group $E_6 \supset SU(3) \times SU(2) \times U(1)$ (which is thus a candidate for a GUT) [20].

There is one more superstring theory,

- **Type I:** Which includes both open and closed strings in $d = 10$. This theory theory has 16 conserved supercharges, which manifest themselves as a spacetime Majorana-Weyl fermion. The massless particle content includes a supergravity multiplet (G, B, and Φ with left handed Majorana-Weyl gravitino and right handed Majorana-Weyl dilatino) in the closed string sector and a super-Yang-Mills multiplet in the open string sector. The particle content is the same as that of the $SO(32)$ heterotic superstring. The type I superstring arises from the type IIB superstring by modding out that theory's left-right symmetry in a procedure called *orientifold projection* (a projection onto strings that are invariant under reversal of orientation) which, among other things, adds the twisted sector of open strings.

5. The View from Spacetime: The Low Energy Supergravity Approximation

In the spirit of section 2, let's finally look at the effective low-energy limits of the various $d = 10$ superstring theories considered in the previous section[31]. Just as the low-energy effective action (7) was essentially a classical gravity + matter action, the superstring low-energy effective actions will be classical *supergravity* + (supersymmetric) matter actions. Since the $d = 10$ superstring theories all have either 16 or 32 supersymmetry generators, the low-energy effective action will turn out to

[31] I guess *we* are the poor, ignorant, spacetime physicist!

be completely determined by the high degree of supersymmetry.

The overarching idea here is that, in the low-energy limit, it is a good approximation to replace string theory by a supergravity theory describing the interactions of the massless modes *only*, since, at weak coupling, the masses of all other states become quite large, and thus too heavy to be observed.

5.1. **The Type IIA Superstring.** Recall the Bosonic components of the action for the $d = 11 \to d = 10$ dimensionally reduced supergravity

$$S_1 = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(e^{\sigma}R - \frac{1}{2}e^{3\sigma}\left|F_2\right|^2\right)$$

$$S_2 = -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(e^{-\sigma}\left|F_3\right|^2 + e^{\sigma}\left|F_4'\right|^2\right)$$

$$S_3 = -\frac{1}{4\kappa_{10}^2}\int A_2\wedge F_4\wedge F_4 = -\frac{1}{4\kappa_{10}^2}\int A_3\wedge F_3\wedge F_4$$

Since we now want to interpret these as having descended from string theory in the low energy limit, we must be able to relate the fields above to the matter content of our string theory via field redefinitions. We *can* do this, be redefining

$$g_{\mu\nu}\to e^{-\sigma}g_{\mu\nu} \quad \text{and} \quad \Phi = \frac{3\sigma}{2}$$

giving

$$S_{\text{IIA, Eff}} = S_{\text{NS, Eff}} + S_{\text{R, Eff}} + S_{\text{CS, Eff}} \tag{16}$$

with

$$S_{\text{NS, Eff}} = \frac{1}{\kappa_{10}^2}\int d^{10}x\sqrt{-g}\,e^{-2\Phi}\left(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|H_3\right|^2\right) \tag{17}$$

$$S_{\text{R, Eff}} = -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-g}\left(\left|F_2\right|^2 + \left|F_4'\right|^2\right) \tag{18}$$

$$S_{\text{CS, Eff}} = -\frac{1}{4\kappa_{10}^2}\int B_2\wedge F_4\wedge F_4 \tag{19}$$

where we have also renamed some of the gauge fields to match the matter content of the string theory. Note that the dilaton dependence has been absorbed into the definition of the fields in the latter two terms, but is definitely still present.

Among other things, this shows that the type IIA string is really the dimensional reduction of an eleven dimensional theory. This eleventh dimension is hidden in perturbation theory but begins to reveal itself at strong coupling. This is the starting point of **M-Theory**[32].

[32]This is also why we expect $d = 11$ supergravity to be the low-energy limit of M-theory.

5.2. **The Type IIB Superstring.** The main problem in constructing the low energy limit of the type IIB theory is that the self dual field strength F_5 prevents us from formulating the low-energy action in a manifestly covariant form. The problem stems from the fact that a term of the form $\int d^{10}x\,|F_5|^2$ does not take the self-duality into account and hence describes twice the number of actual, propagating degrees of freedom. The introduction of a Lagrange multiplier turns out to not help the situation either (the Lagrange multiplier itself ends up reintroducing these same extraneous degrees of freedom). The two usual options here are to either focus instead on the field equations[33] (which *are* manifestly covariant) or to write down a covariant action which needs to be supplemented by a self-duality constraint. There is a more modern third option, the PST (Pasti-Sorokin-Tonin) formalism, which introduces an auxiliary scalar field and a compensating gauge symmetry to give a theory which accounts for the self-duality constraint and is diffeomorphism invariant in all but one directions (up to some topological considerations).

The most remarkable feature of the type IIB SUGRA approximation is an emergent global $SL(2,\mathbb{R})$ symmetry which acts on the two form potentials as a doublet. This has been seen to actually ascend to an exact $SL(2,\mathbb{Z})$ symmetry of the full string theory.

5.3. **The Type I Superstring.** The massless closed string sector of the type I theory descends to $d=10$, $\mathcal{N}=1$ SUGRA and the massless open string sector to $d=10$, $\mathcal{N}=1$ super Yang-Mills with gauge group $SO(32)$. The low energy effective action of the theory (we restrict ourselves to the Bosonic sector for simplicity) simply describes the interaction of these two supermultiplets to leading order in α':

$$S_{\text{I, Eff}} = \frac{1}{2\kappa_{10}^2}\int d^{10}\sqrt{-g}\left(e^{-2\Phi}(R+4\,\partial_\mu\Phi\partial^\mu\Phi) - \frac{1}{2}\left|F_3'\right|^2 - \frac{\kappa^2}{g^2}\,e^{-\Phi}\text{Tr }|F_2|^2\right) \tag{20}$$

where F_2 is the Yang-Mills field strength and F_3' a three form with nonstandard Bianchi identity, built from a type I gauge field and a Chern-Simons term. The first two terms come from a spherical worldsheet ($\chi=-2$) while the last term comes from a disk worldsheet ($\chi=-1$).

The parameter g is related to the $d=10$ Yang-Mills coupling g_{YM} via

$$\frac{g_{\text{YM}}^2}{4\pi} = g_s\frac{g^2}{4\pi} = g_s(2\pi\ell_s)^6$$

This stems from the fact that g_{YM} is an *open* string coupling, and so it is proportional to $\sqrt{g_s}$, which in turn is a consequence of the fact that open strings couple to worldsheet boundaries (while closed strings couple to interior points). We can also figure out low energy contributions beyond $\mathcal{O}(\alpha')$ through a careful study of anomaly cancellation.

5.4. **The Heterotic Superstring.** For the *massless* part of the Heterotic superstring spectrum, the only difference between the $SO(32)$ and $E_8\times E_8$ theory is the exchange of gauge groups. Since there is no vector representation of $E_8\times E_8$, the actions will differ by a normalization, which we can repair by appropriately normalizing the trace of the Yang-Mills field strength in both cases.The low-energy effective action for the heterotic superstring is thus given for *both* gauge groups by

$$S_{\text{Het, Eff}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}e^{-2\Phi}\left(R+4\,\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}\left|H_3'\right|^2 - \frac{\kappa^2}{30g^2}\text{Tr }|F_2|^2\right) \tag{21}$$

this is quite similar to the action for the type I string, the uniform dilaton dependence stemming from the absence of open strings or RR fields. Because of the high degree of supersymmetry, however, this can only differ from the type I action by a field redefinition. If we define

$$G_I = e^{-\Phi_h}G_h,\quad \Phi_I = -\Phi_h,\quad F_{3,I}' = H_{3,h}',\quad \text{and}\quad A_{1,I} = A_{1,h}$$

[33] and corresponding SUSY transformations

we recover the action (20). The only difference is in the relation between κ_{10}, g_{10}, and α'.

5.5. **M-Theory.** While there are no chiral spinors in $d = 11$, $d = 11$ SUGRA allows for chiral matter *within the context of string theory* by exploiting the presence of various Dp-branes. We can also gain emergent chirality through compactification. Since $d = 11$ SUGRA is nonrenormalizable, it is important to try to figure out, and keep track of, which of the infinite number of counterterms are necessary to account for at low energies. This is done by matching $d = 11$ SUGRA to M-theory at low energies by exploiting various dualities between M-theory and the superstring theories. For example, this tells us that in a flat $\mathbb{R}^{10,1}$ background, there are important R^4 terms to account for.

6. Conclusion

We have seen that string theory makes bold predictions for gravity beyond general relativity, while still begging for a deeper development and better understanding of some of its own gravitational aspects. Furthermore, we have seen that the superstring theories, in part through their low energy supergravity approximations, seem to suggest descent from a theory shrouded in mystery—M-theory. These issues—including those of background independence, gravitational string phenomenology, stringy geometry, and the search for M-theory—provide with exciting challenges for the future and guidance in our ongoing struggle to understand Nature.

Appendix A. Supersymmetry

A.1. **The Origins of Supersymmetry.** The Standard Model paints a wildly accurate picture of reality at low energies and vanishing curvature, but it is not perfect. In order to elucidate some of the features of the Standard Model that could not be explained within its own framework, and, even more importantly, to understand features of reality that the Standard Model simply does not explain, Physicists began to look to physics beyond the Standard Model itself.

One way to discover new physics and unify existing theories is by examining the possible symmetries of (the S-matrix of) our universe and then breaking the ones that we don't see. By the 1960's we were aware of two main types of symmetries–*spacetime symmetries* generated by the Poincaré algebra $\mathfrak{so}(1,d-1)\ltimes\mathbb{R}^d$, and *internal symmetries* such as the gauge symmetries of particle physics. We might expect to obtain new symmetries by combining these, but we hit a roadblock: the **Coleman-Mandula Theorem**

Theorem A.1 (Coleman-Mandula)**.** *In a theory of 0-branes with a mass gap and non-trivial scattering in $d > 1+1$, the only possible conserved tensors are the generators P_μ and $J_{\mu\nu}$ of the Poincaré group and scalar internal symmetry charges which commute with P_μ and $J_{\mu\nu}$. In particular, the only possible Lie algebras of symmetries of the S-matrix are direct products of the Poincaré group with internal symmetry groups—we cannot mix the two.*

We may, however, find a loophole. We consider adding new symmetry generators that are, rather than Lie-Algebra valued tensors, *graded* Lie-Algebra valued *spinors*. This is supersymmetry.

A.2. **The Supersymmetry Algebra.** Consider a $\mathbb{Z}_2$-graded algebra $\mathcal{A}=\mathcal{A}_F\oplus\mathcal{A}_B$ where the elements of $\mathcal{A}_F$ have their algebraic properties defined via anticommutators (e.g. like Fermionic fields) while the elements of $\mathcal{A}_B$ have their algebraic properties defined via commutators (e.g. like Bosonic fields). $\mathcal{A}_F$ (resp. $\mathcal{A}_B$) is closed under anticommutation (resp. commutation), whereas the commutator of an element of $\mathcal{A}_F$ with an element of $\mathcal{A}_B$ will yield an element of $\mathcal{A}_F$. $\mathcal{A}_F$ is called the **supersymmetry algebra** and $\mathcal{A}$ is called the **superalgebra**.

The elements of $\mathcal{A}_F$ are operator-valued spinors (self-adjoint since we want a unitary representation) Q^i_α and $\bar{Q}^i_{\dot\alpha}$ $(i=1,\dots\mathcal{N})$ which transform as elements of the $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$ representations of $SL(2,\mathbb{C})$ respectively, e.g.

$$\left[P_\mu,Q^i_\alpha\right]=0=\left[P_\mu,\bar{Q}^i_{\dot\alpha}\right]\qquad\left[J_{\mu\nu},Q^j_\alpha\right]=i(\sigma_{\mu\nu})_\alpha{}^\beta Q^j_\beta\qquad\left[J_{\mu\nu},\bar{Q}^j_{\dot\alpha}\right]=i(\overline{\sigma}_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\,\bar{Q}_{\dot\beta j}$$

Thus the anticommutator of these must transform as an element of the $\left(\frac{1}{2},\frac{1}{2}\right)$ representation of $SL(2,\mathbb{C})$. We guess that

$$\left\{Q^i_\alpha,\bar{Q}^j_{\dot\beta}\right\}=2\,\sigma^\mu_{\alpha\dot\beta}\,P_\mu\,\delta^{ij}\tag{22}$$

since this is the only natural way to use our natural d-vector P_μ and get the correct spinor indices and transformation properties, up to choice of normalization. Note that we may reformulate SUSY in terms of Dirac spinors with e.g.

$$\left\{Q^i_\alpha,\left(Q_{\beta j}\right)^\dagger\right\}=-2\,P_\mu\gamma^\mu_{\alpha\beta}\,\delta^{ij}\gamma^0\tag{23}$$

(the factor of γ^0 appears since we should really be using the Dirac adjoint). For $\mathcal{N}>1$, our theory also includes **central charges** $Z^{[ij]}$ which commute with the elements of the supersymmetry and are defined by

$$\left\{Q^i_\alpha,Q^j_\beta\right\}=\epsilon_{\alpha\beta}Z^{ij}\qquad\left\{\bar{Q}^i_{\dot\alpha},\bar{Q}^j_{\dot\beta}\right\}=\epsilon_{\dot\alpha\dot\beta}\,(Z^{ij})^*$$

This is the **extended SUSY algebra**, which is the most general graded Lie algebra that respects the symmetries of the S-matrix.

A.3. **Supermultiplets.** Basic particle states transform as unitary representations of just the $d+1$ dimensional Poincaré group, which means that they can be intermixed by supersymmetry transformations. Grouping these particles into representations of the whole superalgebra gives us **supermultiplets**.

Recalling that $[Q^A_\alpha, P_\mu] = 0$, we see that the elements of a supermultiplet all share a common mass. They do not, however, all share the same spin—their spins differ by factors of $\frac{1}{2}$ (since the SUSY operators themselves carry spin). We also note that the Hamiltonian is positive definite in supersymmetry, which can be seen by taking the trace of either (22) or (23), giving

$$P^0 = H = \frac{1}{4}\left(QQ^\dagger + Q^\dagger Q\right) \geq 0$$

which in turn implies that

$$\langle\, 0 \,|\, H \,|\, 0 \,\rangle = 0$$

In particular, supersymmetry (should it exist) must be broken in any universe with a nonzero vacuum energy.

Some simple algebra gives

$$\left[J_{12}, Q^A_1\right] = \frac{1}{2}\, Q^A_1 \qquad \left[J_{12}, Q^A_2\right] = -\frac{1}{2}\, Q^A_2$$

and similar for the conjugate spinor operators, so the supersymmetry operators do indeed send Fermions to Bosons and vice versa. We can also prove, using the vanishing of the trace of the operator $(-1)^{N_F}$, that supermultiplets always contain the same number of Fermions as Bosons.

A.3.1. *Massless Supermultiplets.* For massless supermultiplets, we consider the lightlike reference frame $P_\mu = (-E, 0, \dots, 0, E)$. In this frame the algebra becomes

$$\left\{Q^i_\alpha, \bar{Q}^j_{\dot\beta}\right\} = 2 \begin{pmatrix} 2E & 0 \\ 0 & 0 \end{pmatrix} \delta^{ij}$$

We may define creation and annihilation operators

$$a^i \coloneqq \frac{1}{2\sqrt{E}}\, Q^i_1 \qquad a^\dagger_j \coloneqq \frac{1}{2\sqrt{E}}\, \bar{Q}_{\dot 1 j}$$

which lower and raise the helicity respectively with ground state (of helicity λ) given by the **Clifford vacuum** $|\, \Omega \,\rangle_\lambda$, so that all states in the massless Fock space are given as

$$|\, \Omega \,\rangle^{(n)}_{\lambda+\frac{n}{2}, j_1, \dots, j_n} \coloneqq \frac{1}{\sqrt{n!}}\, a^\dagger_{j_n} \dots a^\dagger_{j_1}\ |\, \Omega \,\rangle_\lambda$$

There are $\binom{\mathcal{N}}{n}$ states of each helicity $\lambda - n/2$, and so we have a $2^{\mathcal{N}}$-dimensional representation with $2^{\mathcal{N}-1}$ each of Bosons and Fermions. The highest helicity possible is $\lambda + \mathcal{N}/2$. In order to preserve CPT symmetry, since CPT reverses the helicity, we consider reducible representations given by direct sums

$$\left\{|\, s_1 \,\rangle, \dots, \left|\, s_{\lambda+\mathcal{N}/2} \,\right\rangle\right\} \oplus \left\{\left|\, -s_{\lambda+\mathcal{N}/2} \,\right\rangle, \dots, |\, -s_1 \,\rangle\right\}$$

A.3.2. *Massive Supermultiplets.* For massive supermultiplets of mass M, in the rest frame $P_\mu = (-M, 0, 0, \dots)$, the supersymmetry algebra (in terms of Dirac fermions) takes the form

$$\left\{Q^i_\alpha, Q^\dagger_{\beta j}\right\} = 2M\, \delta_{\alpha\dot\beta}\, \delta^i{}_j$$

$$\left\{Q^i_\alpha, Q^j_\beta\right\} = \epsilon_{\alpha\beta}\, Z^{ij} \qquad \left\{\left(Q^i_\alpha\right)^\dagger, \left(Q^j_\beta\right)^\dagger\right\} = \epsilon_{\alpha\beta}\, (Z^{ij})^*$$

We consider the even case (the odd case is analogous). Since the central charges commute with the elements of the above algebra, we may choose a basis that diagonalizes them, and further apply a unitary transformation that take them to canonical antisymmetric block diagonal form

$$Z^{ij} = \begin{cases} \epsilon \otimes D & \mathcal{N} \text{ even} \\ \begin{pmatrix} \epsilon \otimes D & 0 \\ 0 & 0 \end{pmatrix} & \mathcal{N} \text{ odd} \end{cases}$$

where D is a diagonal matrix with eigenvalues Z_i. We decompose our indices as $i = (a, m)$, $j = (b, n)$ $a, b = 1, 2$ and $m, n = 1, \ldots, \mathcal{N}/2$ (there should be no confusion with worldsheet or Lorentz indices). After our transformations, the algebra takes the from

$$\left\{ Q_\alpha^{an}, \left(Q_\beta^{bm}\right)^\dagger \right\} = 2M\, \delta_{\alpha\beta}\, \delta^{ab}\, \delta^{mn}$$

$$\left\{ Q_\alpha^{am}, Q_\beta^{bn} \right\} = \epsilon_{\alpha\beta}\, \epsilon^{ab}\, \delta^{mn} Z_n \qquad \left\{ (Q_\alpha^{am})^\dagger, \left(Q_\beta^{bn}\right)^\dagger \right\} = \epsilon_{\alpha\beta}\, \epsilon^{ab}\, \delta^{mn} Z_n^*$$

We define annihilation operators

$$a_\alpha^n = \frac{1}{\sqrt{2}} \left(Q_\alpha^{1m} + \epsilon_{\alpha\beta} \left(Q_\beta^{2m}\right)^\dagger \right) \qquad b_\alpha^n = \frac{1}{\sqrt{2}} \left(Q_\alpha^{1m} - \epsilon_{\alpha\beta} \left(Q_\beta^{2m}\right)^\dagger \right)$$

which satisfy the (anti)commutation relations

$$\left\{ a_\alpha^n, \left(a_\beta^m\right)^\dagger \right\} = (2M + Z_n)\delta^{nm}\delta_{\alpha\beta} \qquad \left\{ b_\alpha^n, \left(b_\beta^m\right)^\dagger \right\} = (2m - Z_n)\delta^{nm}\delta_{\alpha\beta}$$

We have that $Z_n \leq 2M$ $\forall n$. If a subset $Z_i = 2M$, $1 \leq i \leq r$, the corresponding b_i operators must vanish, leaving us with a Clifford algebra of $2(N - r)$ creation and annihilation operators. The highest helicity possible is $\lambda + \mathcal{N}$.

A.4. **Superfields and Superspace.** The superspace formalism is a framework for field theory in which supersymmetry is manifest. It will aid us in constructing spinor (and other field) representations of the superalgebra and in defining supersymmetric quantum field theories, in particular the superconformal worldsheet field theories which describe the superstrings. For simplicity, we only consider the case $\mathcal{N} = 1$.

Superspace is an extension of Minkowski space which includes as coordinates internal anticommuting/Fermionic parameters $(\theta_\alpha, \bar{\theta}_{\dot{\alpha}})$ (compare to x^μ which are commuting/Bosonic) so that we may exponentiate our superalgebra to get a Lie group

$$G(x, \theta, \bar{\theta}) = \exp\left[-i\left(x^\mu P_\mu + \theta_\alpha Q_\alpha + \bar{\theta}_{\dot{\alpha}} \bar{Q}_{\dot{\alpha}}\right)\right]$$

Note that superspace has additional tangent vectors

$$\frac{\partial}{\partial \theta^\alpha} \qquad \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}$$

If we analyze an infinitesimal shift by $(0, \xi, \bar{\xi})$, i.e. look the group element $G(0, \xi, \bar{\xi})G(x, \theta, \bar{\theta})$, using the Baker-Campbell-Hausdorff formula, we find that the elements of the supersymmetry algebra manifest themselves as the generators of infinitesimal θ and $\bar{\theta}$ translations in superspace:

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu \qquad \bar{Q}_{\dot{\alpha}} = \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha \sigma^\mu_{\alpha\dot{\beta}} \epsilon^{\dot{\beta}\dot{\alpha}} \partial_\mu$$

Repeating the above for $G(x, \theta, \bar{\theta})G(0, \xi, \bar{\xi})$ gives us another pair of differential operators

$$D_\alpha = -\frac{\partial}{\partial \theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu \qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \tag{24}$$

which obey the algebra

$$\left\{D_\alpha, \bar{D}_{\dot{\alpha}}\right\} = 2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \qquad \{D_\alpha, D_\beta\} = \left\{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\right\} = 0$$

(note that this is the same as the SUSY algebra but with an extra minus sign) and anticommute with the SUSY generators.

A **superfield** is simply an element of a representation of the supersymmetry algebra that is a field over superspace (contrast the situation of a regular quantum field, which is a representation of the Poincaré algebra over Minkowski space). Superfields may be expanded into components as

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \theta\,\psi(x) + \bar{\theta}\,\overline{\chi}(x) + \theta\theta\,m(x) + \bar{\theta}\bar{\theta}\,n(x) + \theta\sigma^\mu\bar{\theta}\,v_\mu(x) + \theta\theta\bar{\theta}\,\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\,\theta\rho(x) + \theta\theta\bar{\theta}\bar{\theta}\,d(x)$$

This works since all other components vanish by the anticommutativity of the superspace coordinates. We may find the transformation properties of the component fields by matching in powers of the superspace coordinates with the transformation of Φ. Note that superfields form reducible representations in general, and so, in order to obtain irreducible representations, we must impose certain constraints to project out some of the components (i.e.), in a way that doesn't yield boundary conditions (which would limit the range of x). It just so happens that we only require two types of superfields (i.e. two types of constraints) to construct all supersymmetric renormalizable Lagrangians: **chiral superfields** given by $\bar{D}\Phi = 0$ and **vector superfields** given by $V = V^\dagger$. These are studied in detail in [19].

Appendix B. Supergravity

B.1. **Why Supergravity?** Supergravity is a theory which unifies general relativity with supersymmetric matter fields. This is in one shot both a theory of physics beyond the standard model, and an extension of general relativity, which is riddled with its own problems (e.g. singularities). Supergravity is a very useful tool in supersymmetric phenomenology (we often wish to consider the MSSM coupled to $\mathcal{N} = 1$ supergravity), a valuable technical tool (it is used, e.g. in Witten's proof of the positive energy theorem), and, as we will see in this paper, a reliable low-energy effecive approximation to the superstring theories and even M-theory. Supergravity grand unified theories also explain the origin of the tachyonic mass term of the Higgs.

By adding more symmetry, there is some hope that supergravity cancels some of the divergences of the naïve linearized gravity theory (there is some belief that $\mathcal{N} = 8$ supergravity may even be finite). Most of all, supergravity is necessary in order to turn supersymmetry into a gauge theory. In fact, supergravity *is* the gauge theory of local supersymmetry. We see this as follows:

Consider two consecutive infinitesimal global supersymmetric transformations of a Boson field $[B] = 1$ with superpartner $[F] = 3/2$:

$$\delta_1 B \sim \bar{\epsilon}_1\, F \qquad \delta_2 F \sim \varepsilon_2\, \partial B$$

by dimensional arguments, where ϵ is an anticommuting Fermionic parameter with $[\epsilon] = -1/2$ and the derivative in the second transformation is required to match dimensionality. This implies, then, that two supersymmetry transformations yield a spacetime translation:

$$\{\delta_1, \delta_2\}B \sim \underbrace{\bar{\epsilon}_2\gamma^\mu\epsilon_1}_{a^\mu}\,\partial_\mu B$$

which makes sense since supersymmetry extends Poincaré symmetry, e.g.

$$\{Q, \bar{Q}\} = 2\sigma^\mu P_\mu$$

Promoting ϵ to a local paramter $\epsilon(x)$ gives us a spacetime dependent translations $a^\mu(x)\partial_\mu$, which we recognize as an infinitesimal diffeomorphism. Thus *local* supersymmetry is simply the gauge

theory of diffeomorphism—but this is gravity.

Supersymmetry, as constructed in Appendix A, is a *global* symmetry. We just found out that if we would like to extend supersymmetry to a *local* symmetry, we *must* incorporate gravity to mediate diffeomorphisms. We consider tackling this problem by extending Einstein gravity to a supermanifold (i.e. curved superspace) with coordinates $z^A = (x^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}})$. Recall that the components of chiral supermultiplets (ϕ_i, ψ_i, F_i) (where ϕ_i is a scalar, ψ_i Weyl spinors, and F_i auxiliary scalars) are obtained as coefficients in the expansion of our superfields $\Phi_i(x, \theta, \bar{\theta})$. We obtain a new metric g_{AB} which is invariant under *superspace diffeomorphisms*

$$z^A \to z^A + \xi^A(z)$$

(note that this includes both spacetime diffeomorphisms and and local supersymmetry transformations) and contains both the vielbeins and a spin-3/2 field called the **gravitino** (the superpartner of the graviton). An excellent coverage of geometry on supermanifolds can be found in [12].

B.2. **An Example: $d = 4$, $\mathcal{N} = 1$ Supergravity.** This is the simplest example of a supergravity theory, but also has use in practice, as the simplest way to couple the (Minimally Supersymmetric) Standard Model to gravity (besides, of course, the most naïve way). This will give us an area to develop important terminology and see some general features in action.

In 4d superspace, the most general renormalizable (globally) supersymmetric Lagrangian involving only chiral superfields (ignoring linear terms since these will be forbidden by gauge invariance for non-neutral fields) is given by

$$\mathscr{L}_{\text{global}} = \int d^2\theta\, d^2\bar{\theta}\, \Phi_i^\dagger \Phi_i + \left[\int d^2\theta \left(\frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} Y_{ijk} \Phi_i \Phi_j \Phi_k \right) + \text{h.c.} \right] \tag{25}$$

Where m and Y are totally symmetric in their indices. We rewrite this for future use as

$$\mathscr{L}_{\text{global}} = \int d^2\theta \left[-\frac{1}{8} \bar{D}\bar{D}\, \Phi_i^\dagger \Phi_i + \frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} Y_{ijk} \Phi_i \Phi_j \Phi_k \right] + \text{h.c.} \tag{26}$$

where $\bar{D}$ is given by (24) and we have used the identity

$$\int F(x, \theta, \bar{\theta})\, d^2\theta\, d^2\bar{\theta} = -\frac{1}{4} \int \bar{D}\bar{D} F(x, \theta, \bar{\theta})\, d^2\theta$$

To generalize this to the case of local supersymmetry mediated by supergravity, we first allow our coordinates θ_α to generalize to tangent space coordinates $\Theta_\alpha^{a\cdots}$ (since we are adding curvature to our supermanifold, it is nicer to retain our old machinery and simply work in an $SL(2, \mathbb{C}) \ltimes \mathbb{R}^4$-associated $\mathbb{R}^{3,1+4}$ superspace vector bundle generalizing the local Lorentz bundle of the previous appendix (see Appendix C) and introduce an analog of the gravitational action eq. (30). We need a supersymmetric generalization of the measure

$$|\text{vol}| = \left| e^0 \wedge e^1 \wedge e^2 \wedge e^3 \right| = e\, d^4x$$

(where we have defined $e \coloneqq \det(e^a{}_\mu) = \sqrt{-g}$) as well as the Ricci scalar R. These are given by the chiral density superfield $\mathcal{E}$ (which generalizes e) and the superspace curvature superfield $\mathcal{R}$ (which generalizes R) with components

$$\mathcal{E} = (e, \Psi_\mu, \dots) \qquad \mathcal{R} = (R, \Psi_\mu, \dots)$$

where Ψ_μ is the **gravitino**, the spin-3/2 superpartner of the spin-2 gravition. The dots denote auxiliary fields that arise as coefficients of the expansion of the superfields in powers of Θ and $\bar{\Theta}$.

The Lagrangian for the free supergravity field is thus given by

$$\mathscr{L}_{\text{sg}} = -6 \int d^2\Theta \, \mathcal{E} \, \mathcal{R}$$

Where the essential changes came from replacing coordinates $\theta \to \Theta$ and generalizing the volume form $d^2\theta \to d^2\Theta \, 2\,\mathcal{E}$. We now couple our superfields to gravity by sending

$$-\frac{1}{4}\bar{D}\bar{D} \to -\frac{1}{4}\left(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8\mathcal{R}\right)$$

where $\mathcal{D}$ and $\bar{\mathcal{D}}$ are obtained from (24) via minimal substitution $\partial_\mu \to \hat{\nabla}_\mu$ where $\hat{\nabla}$ is the supersymmetric covariant derivative (so called because it is a function of the supersymmetric spin connection, which in turn is now a function of the entire supergravity multiplet rather than just the vielbein). Generalizing the global chiral Lagrangian (26) and adding the pure supergravity Lagrangian we get

$$\mathscr{L}_{\text{local}} = \int d^2\Theta \, 2\mathcal{E} \left[-\frac{1}{8}\left(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8\mathcal{R}\right)\Omega(\Phi_i, \Phi_i^\dagger) + W(\Phi_i)\right] + \text{h.c.} \tag{27}$$

where

$$\Omega(\Phi_i, \Phi_i^\dagger) \coloneqq \Phi_i^\dagger \Phi_i - 3$$

is the superspace kinetic energy and

$$W(\Phi_i) = \frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} Y_{ijk} \Phi_i \Phi_j \Phi_k$$

the **superpotential**. After eliminating non-dynamical auxiliary fields and some rescaling and redefinition, we may define a component Lagrangian in terms of the physical fields ϕ, ψ, $e^m{}_\mu$, and Ψ_μ and a real function, the **Kähler potential**

$$K(f, f^*) \coloneqq -3 \ln\left(-\frac{\Omega(f, f^*)}{3}\right)$$

from which we can extract e.g. the kinetic energies of the scalars as

$$\frac{\partial^2 K}{\partial \phi_i \, \partial \phi_j^*} \partial_\mu \phi_i \, \partial^\mu \phi_j^*$$

The Lagrangian has many higher order interaction terms, including nonrenormalizable terms. This Lagrangian has a natural interpretation in the language of Kähler geometry which is described in Appendix A of [5].

We finish up by noting that, as supergravity is nonrenormalizable, we should interest ourselves with starting not just from the most general *renormalizable* global chiral Lagrangian (25), but the most general *possible* Lagrangian that can be built from chiral superfields

$$\mathscr{L}_{\text{global, non-renormalizable}} = \int d^2\theta \, d^2\bar{\theta} \, K(\Phi_i, \Phi_j^\dagger) + \left[\, d^2\theta \, W(\Phi_i) + \text{h.c.}\right] \tag{28}$$

where now K and W are arbitrary vector and chiral superfields respectively, with power series expansions in terms of the chiral superfields Φ_i.

B.2.1. *Coupling to Matter.* We want to obtain a complete theory of supersymmetric chiral matter coupled to both pure supergravity and Yang-Mills internal gauge fields.

Let us forget about gravity and local supersymmetry briefly to consider the construction of a renormalizable gauge invariant supersymmetric Lagrangian. We impose gauge invariance on the general renormalizable (globally) supersymmetric chiral Lagrangian (25) by introducing a vector superfield $V \coloneqq 2g_i T^i V^j$, where T^i are the generators of the gauge group G, V^j the gauge supervector fields (so j is here a $\mathfrak{g}$ Lie algebra index), and g_i the couplings, via

$$\mathscr{L}_{\text{global, gauge}} = \int d^2\theta\, d^2\bar{\theta}\, \Phi_i^\dagger \exp[V]\, \Phi_i + \left[\int d^2\theta \left(\frac{1}{2} m_{ij}\Phi_i\Phi_j + \frac{1}{3} Y_{ijk}\Phi_i\Phi_j\Phi_k\right) + \text{h.c.}\right]$$

where the Φ_i now transform as (possibly different) representations of G and the components of V^i include the gauge vector Bosons (transforming in the adjoint representation), their Majorana spinor superpartners, and auxiliary scalar fields. We define the (chiral spinor) gauge field strength superfield

$$F_\alpha = -\frac{1}{4}\bar{D}\bar{D}\, \exp[-V]\, D_\alpha\, \exp[V]$$

so that the Yang-Mills kinetic term is given by

$$\mathscr{L}_{YM} = \frac{1}{16}\int d^2\theta\, F^{i\alpha} F_{i\alpha} + \text{h.c.}$$

Since we again disregard renormalizibility as soon as we incorporate supergravity, we can generalize above renormalizable super Yang-Mills theory to a nonrenormalizable super Yang-Mills theory by analogy with (28):

$$\begin{aligned}\mathscr{L}_{\text{YM, non-renormalizable}} = &\int d^2\theta\, d^2\bar{\theta}\left[K(\Phi_i, \Phi_j^\dagger) + C(\Phi_i, \Phi_j^\dagger, V)\right] + \left[\int d^2\theta\, W(\Phi_i) + \text{h.c.}\right] \\ &+ \frac{1}{16}\int d^2\theta\, f_{jk}(\Phi_i)\, F^j F^k\end{aligned}$$

where C is a counterterm necessary for maintaining gauge invariance and f_{ij} an arbitrary analytic function of the chiral superfields (which would be just δ_{ij} in the renormalizable case). Finally, sending $F_\alpha \to \mathcal{F}_\alpha$, the curved-space field strength superfield given by

$$\mathcal{F}_\alpha = -\frac{1}{4}\left(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8\mathcal{R}\right)\, \exp[-V]\, \mathcal{D}_\alpha \exp[V]$$

we obtain the $d = 4$, $\mathcal{N} = 1$ **supergravity Yang Mills Lagrangian**

$$\begin{aligned}&\mathscr{L}_{\text{SGYM}} \\ &= \int d^2\Theta\, 2\mathcal{E}\left[\frac{3}{8}\left(\bar{\mathcal{D}}\bar{\mathcal{D}} - 8\mathcal{R}\right)\exp\left\{-\frac{1}{3}\left[K(\Phi_i, \Phi_j^\dagger) + C(\Phi_i, \Phi_j^\dagger, V)\right]\right\} + \frac{1}{16} f_{jk}(\Phi_i)\, F^j F^k + W(\Phi_i)\right] + \text{h.c.}\end{aligned}$$

Letting $G = SU(3)_c \times SU(2)_L \times U(1)_Y$ and letting the matter content be that of the MSSM gives us the simplest way to look at gravity coupled to matter as we know it.

Appendix C. Einstein Meets Dirac: Vielbeins and The Spin Connection

This appendix is purely a fun aside about the technicalities that are involved with coupling Fermions to gravity. This is an important issue, not just because it is essential to string theory, but even classically!

The gauge group of general relativity is the group of all diffeomorphisms of the spacetime manifold. In the tangent bundle (from which we construct all the tensors of our theory), this manifests itself as the general linear group $GL(d, \mathbb{R})$, which does not permit a spinor representation. However

something must be done about this—any observer carries with her a *local* Lorentz frame (which essentially manifests the idea that the *tangent* space at each point is just flat Minkowski spacetimes) and should locally see flat spacetime physics, including spinors. To make this idea precise, we consider (where defined) an orthonormal basis of sections of the tangent bundle (doing this along a particular worldline is what we mean by an observer's local Lorentz frame), the **vielbeins** (German for "many legs" since there are two relevant types of indices) e_m, defined by

$$g_{\mu\nu}\, e_m{}^{\mu} e_n{}^{\nu} = \eta_{mn} \qquad e_m = e_m{}^{\mu}\, \frac{\partial}{\partial x^{\mu}} \tag{29}$$

(note that Greek indices are *spacetime indices*, i.e. indices associated to a coordinate basis that are acted on by general coordinate transformations, whereas the Latin indices are *frame indices* which are acted upon by local Lorentz transformations), give us a Minkowski-vector bundle structure which can be associated to an $O(d-1,1)$ principle bundle (i.e. the Latin indices transform under $O(3,1)$). We may then double-cover $O(d-1,1)$ by Spin(d-1,1) (e.g. for $O(3,1)$, by $SL(2,\mathbb{C})$) to recover spinor fields *in the tangent space* as projective unitary representations! Manifolds where we can do this globally are said to admit a *spin structure*.

Raising the Lorentz indices on both sides of (29), we find that

$$e^m{}_{\nu} e_n{}^{\nu} = \delta^m{}_n$$

which tells us the important fact that the dual basis of the vielbeins is simply the vielbeins with the indices switched up/down. In particular we get that

$$g_{\mu\nu} = e^m{}_{\mu} e^n{}_{\nu}\, \eta_{mn}$$

which is why they are sometimes called the "square root of the metric".

Note that we can translate vector components between these bases via

$$V^{\mu}\frac{\partial}{\partial x^{\mu}} = V^m e_m = V^m e_a{}^{\mu}\frac{\partial}{\partial x^{\mu}} \implies V^{\mu} = V^m e_a{}^{\mu} \implies V^m = e^m{}_{\mu} V^{\mu}$$

The way we couple spinors to gravity is by the **spin connection** $\omega_{\mu}{}^m{}_n$, defined by

$$\nabla_{\mu} e^m{}_{\nu} = \omega_{\mu}{}^m{}_n\, e^n{}_{\nu} \quad \text{i.e.} \quad \nabla e^m = \omega^m{}_n \wedge e^n$$

In the same way that the Christoffel connection is the gauge field corresponding to diffeomorphisms of the spacetime manifold, we may view ω as the gauge field corresponding to local Lorentz transformations $\Lambda^m{}_n(x)$ in the *adjoint representation* acting on the tangent bundle (note that $\omega_{mn} = -\omega_{nm}$ since we are representing local Lorentz transformations, which gives us metric compatibility for free). Recalling that[34] $\Sigma_{mn} \coloneqq \frac{1}{4}[\gamma_m, \gamma_n]$ is the generator of the Lorentz group in the Dirac spinor representation, and using the standard covariant derivative expansion $D_{\mu}\psi = \partial_{\mu}\psi + A^a_{\mu} T_a \psi$, we must have that, for ∇ acting on a (Dirac) spinor field ψ

$$\nabla_{\mu}\psi = \partial_{\mu}\psi + \frac{1}{2}\,\omega_{\mu}{}^{mn}\,\Sigma_{mn}\psi$$

Since the vielbein already contain all the degrees of freedom of gravity (see e.g. [11]), we see that ω must be completely constrained in form by the other ingredients of our theory. We fix the spin connection by asserting that the vielbein have vanishing torsion, defined here via

$$T^m = de^m + \nabla e^m = De^a - (\text{contributions from matter gauge fields})$$

Note that this recovers standard general relativity from the more general Einstein-Cartan theory (GR with nonvanishing torsion). It is interesting to note that one can consider the vielbeins as

[34] Here we are using the normalization of the generators such that an identity component Lorentz transformation is written as $\psi \to \exp\left[\frac{1}{2}\lambda_{mn}\Sigma^{mn}\right]\psi$.

the gauge field corresponding to local translations and the torsion as their field strength; general relativity is then seen to be the special case where the local translation gauge field is flat.

The field strength of the spin connection is the "flattened" Riemann tensor

$$R^m{}_n = d\omega^m{}_n + \omega^m{}_l \wedge \omega^l{}_n$$
$$\implies R_{\mu\nu}{}^m{}_n = \partial_\mu \omega_\nu{}^m{}_n - \partial_\nu \omega_\mu{}^m{}_n + \omega_\mu{}^{ml}\omega_{\nu\,n}{}^{l} - \omega_\nu{}^m{}_l \omega_\mu{}^l{}_n = R_{\mu\nu}{}^\rho{}_\sigma e^m{}_\rho\, e^\sigma{}_n$$

Note that both $\omega^a{}_b$ and $R^a{}_b$ take values in $\mathfrak{so}(3,1)$.

The spin connection also gives us a representation of the covariant derivative on the "frame" representation of our vector fields:

$$\nabla_\mu V^m = \partial_\mu V^m + \omega_\mu{}^m{}_n V^n$$

and, in general, the spacetime covariant derivative acts via the formula

$$(\nabla_\mu \Phi)^i = \partial_\mu \Phi^i + \frac{1}{2}\,\omega_\mu{}^{mn}\,(\Sigma_{mn})^i{}_j \Phi^j$$

where $(\Sigma_{mn})^i{}_j$ is the appropriate representation of the appropriate Lorentz group such that, under an identity component transformation, $\Phi \to \exp\left[\frac{1}{2}\lambda_{mn}\Sigma^{mn}\right]\Phi$ (in analogy with the spinor).

As a final point, note that in terms of the vielbein, the Einstein-Hilbert action is given by

$$S_{\text{EH}} = \frac{1}{16\pi G}\int_M \underbrace{R_{\mu\nu}{}^m{}_n\, e^\mu{}_m\, \eta^{nl}\, e^\nu{}_l}_{R}\ \overbrace{e^0 \wedge e^1 \wedge e^2 \wedge \cdots \wedge e^{d-1}}^{\text{vol}} \tag{30}$$

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