Higher-form Symmetries Exposition

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This paper represents my attempt at making sense of the papers "Higher-form symmetries and spontaneous symmetry breaking" by Ethan Lake and "Generalized Global Symmetries" by Davide Gaiotto, Anton Kapustin, Nathan Seiberg, and Brian Willett.

1 Motivation

Higher-form symmetries are symmetries for which the charged objects are extended. A p-form symmetry has p-dimensional charged objects- ordinary symmetries have charged particles, p = 0. My own motivation (as someone interested in CMT) for figuring out what these are about is that its proponents claim that topological phases may be understood via a symmetry-breaking paradigm in which the symmetries are higher-form. These papers are pretty advanced for my background so I wasn't able to fully grasp their arguments at an intuitive level, so instead I'll go through some of the basics of what higher-form symmetries are and a couple of classic theorems lifted to higher forms.

2 Higher-form Symmetries: What are they?

One way you could express a higher-form symmetry is in the standard way that we express symmetries of an action S. Say we have a p-form field ϕ; we could produce an action (for concreteness, suppose)

$$S = \int \mathrm{d}\phi \wedge \star \mathrm{d}\phi$$

and say that any transformation $\phi \to \phi'$ which leaves the equations of motion undisturbed is a symmetry. But the way that we approached the problem of producing Poincaré-invariant actions in the beginning of the course involved using knowledge about the symmetry group directly, so we should first find a way to express the action of a symmetry group G on a field theory. Let's start by expressing a 0-form symmetry in a manner generalizable to the p-form case.

It turns out the best way to do this is by employing Noether's theorem: associated to any generator of a continuous symmetry there is a a $d-1$-form conserved current j. Integrating over this form on a codimension 1 manifold M yields the conserved charged

$$Q(M) = \oint_M j.$$

We express the symmetry transformations operators associated with a manifold and a group elements, $U_g(M)$, obtained in the case of a continuous symmetry by exponentiating the charge associated with j and M. That the symmetry group elements are *group* elements indicate that these operators should obey a fusion rule:

$$U_g(M)U_{g'}(M) = U_{g''}(M)$$

when $gg' = g''$. Furthermore the dependence of U_g on M is only topological: the value of U_g only changes under deformations of M when M crosses a charged object.

Everything above is slightly more fancy language for the pedestrian Gauss's law: the charge associated to any conserved current contained in some closed boundary is accountable entirely through measuring how much goes in and out of the boundary; i.e., it's conserved. Nothing fancy or nontautological.

So how does it go through when we promote our zero-dimensional objects into extended objects? The intuition is that a p-form symmetry acts on operators supported on p-dimensional manifolds. Furthermore a symmetry with a nonanomalyous p-form symmetry can be coupled to a background $p+1$-form connection, just like how we can couple a 0-form $U(1)$ gauge theory to the 1-form vector potential A_μ.

For a p-form symmetry, Noether's theorem now provides us a closed $d-q-1$-form current j to be integrated over codimension $p+1$ manifolds M, obtaining $Q(M) = \oint_M j$ and $U_q(M)$ exactly as before. We institute the fusion rule as before for $g \in G$, G our symmetry group.

We actually find something interesting about higher symmetries instantly: since any charge operator is supported on a manifold of codimension > 1, any two charge operators can be smoothly deformed so that $Q_1 Q_2 = Q_2 Q_1$ on a space of trivial topology. Hence the symmetry group G for a higher symmetry can be nonabelian only if the underlying space has nontrivial topology.

3 The Higher Nambu-Goldstone Theorem

Let's say we have a p-form symmetry from which we obtain the current j. We can construct the charge operator

$$Q(M) = \int_M \star j.$$

And let $\mathcal{O}$ be a operator charged under a global p-form symmetry which is spontaneously broken, ergo let $\mathcal{O}$ have a nonzero vacuum expectation value $\langle 0|\mathcal{O}|0\rangle \neq 0$.

Consider

$$\mathcal{C} = \langle 0|[Q(M), \mathcal{O}]|0\rangle.$$

We want to choose M in such a way as to make $\mathcal{C}$ nonzero. To do this there is a trick: let $M \subset \Sigma$ where Σ is codimension 1 and let $\hat{M}$ be the Poincaré dual of M with respect to Σ. Then

$$Q(M) = \int_\Sigma \star j \wedge \hat{M}.$$

On Σ, $\star j \wedge \hat{M}$ is just a function (i.e. a 0-form) so we can proceed with a calculation of $\mathcal{C}$ as if it's a typical calculation in a Hilbert space:

$$\begin{aligned}\mathcal{C} &= \sum_n \int_\Sigma \mathrm{d}^{d-1}x \left(\langle 0|(\star j \wedge \hat{M})|n\rangle\langle n|\mathcal{O}|0\rangle - \langle 0|\mathcal{O}|n\rangle\langle n|(\star j \wedge \hat{M})|0\rangle \right) \\ &= \sum_n (2\pi)^{d-1}\delta^{d-1}(\mathbf{p}) \left(\langle 0|(\star j \wedge \hat{M})|n\rangle\langle n|\mathcal{O}|0\rangle \mathrm{e}^{-i\omega_n t} - \langle 0|\mathcal{O}|n\rangle\langle n|(\star j \wedge \hat{M})|0\rangle \mathrm{e}^{i\omega_n t} \right),\end{aligned}$$

letting $t = x^0$ point in the direction normal to Σ.

Our goal is to demonstrate the existence of Goldstone modes. I.e, we'd like to show that the spectrum is gapless at zero momentum, or that we have states satisfying $\omega_n = 0$ when $\mathbf{p} = 0$.

To this end, we'll show that the right-hand side of the equation above, despite carrying explicit time dependence, is independent of t.

We differentiate with respect to t and use Poincaré duality to turn the wedge product with $\hat{M}$ into an integral over M:

$$\partial_t \mathcal{C} = \int_M \langle 0 | [\partial_t \star j, \mathcal{O}] | 0 \rangle.$$

Current conservation imposes the conservation equation on j that $\partial_t j + \mathrm{div} j = 0$, up to some fancy indices since j is a form. Equivalently, in fancy terms,

$$\partial_t \mathcal{C} = -\int_{\partial M \subset \partial \Sigma} \langle 0 | [\star_\Sigma J, \mathcal{O}] | 0 \rangle.$$

We can choose $\mathcal{O}$ such that it is supported only off ∂M, making the integral above vanish entirely.

So we can choose $\mathcal{C}$ to be nonzero such that $\partial_t \mathcal{C}$ vanishes even when it has explicit time dependence in the exponential factors. Our spectrum must then be gapless at zero momentum ($\omega_n = 0$) and we have Goldstone modes, as desired. But what are the Goldstone modes?

The Goldstone modes are those fields which shift by the addition of a "constant" factor under the action of the symmetry group. For $p = 0$, this is the shift $\phi \to \phi + c$. For $p > 0$, this is $A \to A + \lambda$ for λ a flat connection.

Something pretty cool about the earlier fact that higher-form symmetries must all be Abelian is that it greatly simplifies the kinds of infrared effective actions we can write down if the Goldstones are the only massless fields. All we can do is write the kinetic (Maxwell) action

$$S = -\frac{1}{2g^2} \int F \wedge \star F$$

for which $F = \mathrm{d}A$.

(NB: Under some circumstances can write a Chern-Simons or θ term.)

4 The Higher Coleman-Mermin-Wagner Theorem

Recall that in quantum statistical mechanics, continuous symmetries cannot be spontaneously broken at finite temperature for short-ranged systems in dimensions less than 2. The intuition is that the resulting massless Goldstone bosons will have infrared divergence in their correlation functions when $d \leq 2$.

Here we'll prove that continuous p-form symmetries in d dimensions are not spontaneously broken for $p \geq d - 2$.

The scheme is to calculate the the expectation values of Wilson membranes

$$W_C = \exp\left(i \int_C A \right)$$

charged under p-form symmetries. We'll find that these expectation values have log IR diveregence for $p \geq d - 2$.

First we write

$$W_C = \exp\left(i \int_X A \wedge \hat{C} \right)$$

from which, after some heavy-lifting involving and ghosts the expectation value under our previous Maxwell action will be given by

$$\langle W_C \rangle = \int_X \mathcal{D}A\, \mathcal{D}\mu_{\text{ghosts}} \exp\left[-\left(\int_X \frac{1}{2g^2} F \wedge \star F - i\, A \wedge \hat{C}\right) - S_{\text{gauge}} - S_{\text{ghosts}}\right]$$

where S_{gauge} is a gauge-fixing term and S_{ghosts} is a ghost term Furthermore we can write $A = A_{\text{classical}} + A_q$, where $A_{\text{classical}}$ satisfies the equations of motion $\mathrm{d}^\dagger \mathrm{d} A_{\text{classical}} = 0$ and the fields satisfy the boundary conditions

$$A_{\text{classical}}|\partial X = A|\partial X, \qquad A_q|\partial X = 0.$$

Imposing these conditions on the field $F = \mathrm{d}A$ lets it all shake out to (sans ghosts)

$$S = S[A_{\text{classical}}] + \frac{1}{2g^2} \int_X A_q \wedge \star \mathrm{d}^\dagger \mathrm{d} A_q.$$

This can be further gauge fixed by averaging over all co-exact f such that $\mathrm{d}^\dagger A - f = 0$ to obtain

$$S = S[A_{\text{classical}}] + \frac{1}{2g^2} \int_X A_q \wedge \star \left(\mathrm{d}^\dagger \mathrm{d} + \frac{1}{\alpha} \mathrm{d}\mathrm{d}^\dagger\right) A_q$$

where from now on we'll let $\left(\mathrm{d}^\dagger \mathrm{d} + \frac{1}{\alpha}\mathrm{d}\mathrm{d}^\dagger\right) = \Delta$ for brevity.

Going back to our expectation value, we can now write

$$\langle W_C \rangle \sim \int \mathrm{D}A\, \mathrm{e}^{-S[A_{\text{classical}}]} \exp\left[-\int_X A_q \wedge \left(\frac{1}{2g^2} \star \Delta A_q - i\hat{C}\right)\right].$$

The term $\mathrm{e}^{-S[A_{\text{classical}}]}$ is contant with respect to C so we can ignore it. We can also eliminate the $A_q \wedge \hat{C}$ coupling with the gauge transformation

$$A_q \to A_q - ig^2 \Delta^{-1} \star \hat{C}.$$

Effecting this transformation pulls out a Gaussian integral on A_q which resolves to $1/\sqrt{\det(\mathrm{d}^\dagger \mathrm{d} + \mathrm{d}\mathrm{d}^\dagger)}$. Like $\mathrm{e}^{-S[A_{\text{classical}}]}$, this doesn't depend on C so we can again ignore it to write

$$\langle W_C \rangle \sim \exp\left(-\frac{1}{2} g^2 \int \hat{C} \wedge \star \Delta^{-1} \hat{C}\right).$$

Calling Δ^{-1} the "propagator" D as well as choosing $\alpha = 1$, we can squint a little to see that

$$D(k) \sim \frac{1}{k^2}.$$

Now let's take C to be a copy of $\mathbf{R}^p$ embedded in $\mathbf{R}^d$. As usual the universe is just a box with side length L getting really big.

$$\begin{aligned}
\langle W_C \rangle &\sim \exp\left(-\frac{1}{2} g^2 \int_C \int_C \mathrm{d}^p x \wedge \mathrm{d}^p y\, D(x-y)\right) \\
&= \exp\left(-\frac{1}{2} g^2 L^p \int_C \mathrm{d}^p x \int \frac{\mathrm{d}^d k}{(2\pi)^d} \mathrm{e}^{ikx} \frac{1}{k^2}\right) \\
&= \exp\left(-\frac{1}{2} g^2 L^p \int \frac{\mathrm{d}^{d-p} k}{(2\pi)^{d-p}} \frac{1}{k_\perp^2}\right).
\end{aligned}$$

The longitudinal modes *along* the brane are integrated out and do not matter for symmetry breaking considerations. The renormalization

$$W \to W \exp\left(ic \int_C \mathrm{d}^p x\right)$$

will regulate the integral in the UV with some appropriate momentum cutoff. However, the IR log-divergence of the integral above persists for all $p \geq d-2$ and tells us that for such theories $\langle W_C \rangle \to 0$ as $L \to \infty$. Ergo, the symmetry is unbroken.