

Resonant Tunneling in Cosmic Landscape

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Abstract

The resonant tunneling plays an important role in the cosmic landscape and it serves an interesting mechanism to explain why the cosmological constant is such small. This effect has also been applied in condensed matter physics and other fields. Firstly, we will give a brief introduction of Coleman's instanton method on the description of vacuum decay and this is an equivalent description of the functional schrodinger equation method later. Secondly, we will derive the functional schrodinger equation(Or Wheeler-DeWitt equation) method in flat spacetime by simple canonical quantization. Thirdly, we will reduce the whole calculation of field theory to 1D problem by the calculation of MPEP(Most probable escape path). Finally, we will discuss the resonant tunneling in both QM and scalar QFT and show that in multi-barrier situation and when some particular condition is satisfied, the tunneling rate is exponentially enhanced compared to single barrier situation.

1 Introduction

Resonant tunneling is a famous and common phenomenon in quantum mechanics which has similar mechanism as Fabry-Perot interferometer. The resonant tunneling in quantum mechanics [1] can be extended to quantum field theory which has infinite degree of freedoms through Wheeler-DeWitt equation in flat spacetime limit [2] [3] [4] [5] .

The application of the resonant tunneling in QFT is wide. The first application of resonant tunneling in cosmology is put out by Henry Tye in 2006 to give a dynamical explanation in string cosmology on why the cosmological constant is so small [6]. It is also used in the explanation of the 1st order phase transition between A and B phase in superfluid He-3 [7].

To simplify the calculation, we need to find its MPEP(Most probable escape path) to reduce the tunneling calculation in infinite dimension functional space to 1D space. Comparing to Coleman's instanton, the calculation through Wheeler-DeWitt equation has its own advantages even though the calculation might be more complicated. The first advantage is that Wheeler-DeWitt equation can take the multi-barrier situations which are common in cosmic landscape into consideration, while Coleman's instanton calculation can only be used in single barrier situation. The second advantage is that the calculation in Wheeler-DeWitt equation formalism is equivalent to the calculation in Coleman's instanton calculation, not only in thin-wall limit but also in thick-wall limit[4] [8]. Through resonant tunneling, the tunneling rate can be exponentially enhanced comparing to single barrier tunneling so more parameter space of some inflation model with multi-vacuum can be constrained in theoretical way.

2 Coleman's Instanton Method

In 1977, Sydney Coleman put out an Euclidean instanton formalism which describe the quantum tunneling from false vacuum to true vacuum[9] [10]. The gravitational correction of vacuum decay is worked out by Sydney Coleman and Frank De Luccia in 1980 [11]. Coleman's instanton method is sometimes used to constrain the parameter space of some inflation model such as axion monodromy [12]. The calculation of Coleman's instanton is relatively easy compared to the functional Schrodinger method but it can only be used in single barrier case because it cannot take the phase factor in classical allowed region into consideration. For the scalar field with action

$$S = \int d^D x \mathcal{L} = \int d^D x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V[\phi] \right) \quad (1)$$

We can do the Wick rotation and have the Lagrangian in Euclidean space which is

$$S_E = \int d^D x \mathcal{L}_E = \int d^D x \left(\frac{1}{2} \left(\frac{d\phi}{d\tau} \right)^2 + \frac{1}{2} (\nabla \phi)^2 + V[\phi] \right) \quad (2)$$

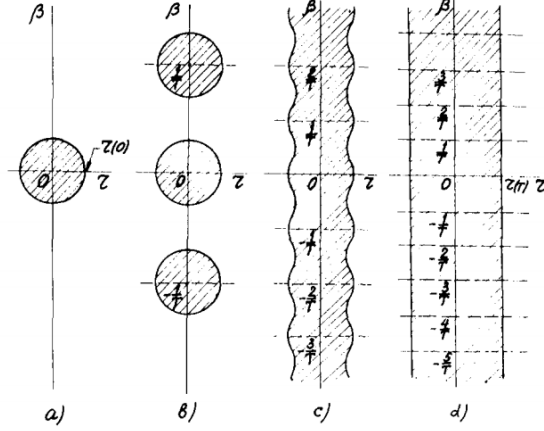


Figure 1: (a) is zero temperature situation which means $T = 0$, the bounce solution satisfies $SO(4)$ symmetry. (b) means $T \ll \frac{1}{R}$, the $SO(4)$ symmetry of bounce solution is slightly broken. (c) means $T \sim \frac{1}{R}$ and the bounce solution doesn't satisfy $SO(4)$ symmetry. (d) means $T \gg \frac{1}{R}$ which is high temperature limit and now the bounce solution satisfies $SO(3)$ symmetry.

Here the potential is that

$$V[\phi] = \frac{g}{4}(\phi^2 - c^2)^2 + (-1)B(\phi + c) \quad (3)$$

The term $(-1)B(\phi + c)$ breaks Z_2 symmetry. To do thin wall approximation later, here $Bc \ll 1$. Using $SO(D)$ ansatz we can derive the EOM with boundary conditions

$$\frac{d^2\phi(r)}{dr^2} + \frac{D-1}{r} \frac{d\phi}{dr} = \frac{\delta U}{\delta \phi} \quad (4)$$

$$\left. \frac{d\phi}{dr} \right|_{r=0} = 0, \quad \phi \Big|_{r \rightarrow +\infty} = -c \quad (5)$$

We can solve this boundary value problem using numerical method such as Newton-Kantorovich method[13] [14] [15]. To get analytical consequence, we can do thin-wall approximation and we can also solve bounce solution which is

$$\phi_B(\tau, \vec{x}) = (-c) \tanh \left[\frac{\sqrt{2gc^2}}{2} (r - R) \right] \quad (6)$$

where $r = \sqrt{\tau^2 + |\vec{x}|^2}$. Now R is still not determined, using the variation principle of S_E we can have that

$$R = \frac{3S_E}{\varepsilon} \quad (7)$$

and

$$S_E = \frac{27\pi^2}{2} \frac{S_1^4}{\varepsilon^3} \quad (8)$$

The whole tunneling probability is

$$\frac{\Gamma}{V} = Ae^{-B_E} \quad (9)$$

and the tunneling probability is exponentially suppressed by the bounce action.

In the application of string cosmology during inflation, since before reheating most of the models are in zero temperature, we choose $D = 4$. In the application of particle cosmology during EW phase transition or QCD phase transition, since the finite temperature will wrap the time dimension and breaks $SO(4)$ symmetry [16], there is $SO(3)$ symmetry and we need to choose $D = 3$.

3 Functional Schrodinger Equation Method

3.1 Curve in Configuration Space

The curve in the configuration space is defined as $\phi|_\lambda = \{\phi(\vec{x}_1; \lambda), \phi(\vec{x}_2; \lambda), \dots, \phi(\vec{x}_N; \lambda)\}$, here $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N$ are treated as continuous indices and λ is treated as the single parameter which label the movement along the curve. The line element is defined as $ds^2 = \int d^3\vec{x} (\delta\phi(\vec{x}))^2$ so we can choose $\lambda = s$ to reparameterize the curve.

Doing the analogy with the curve theory in classical differential geometry, we can define the tangent vector of the curve as $\frac{\partial\phi|_s}{\partial s}$. We can also define the vectors on the cotangent space,

$$\delta_{\parallel}\phi = ds\left(\frac{\partial\phi|_s}{\partial s}\right) \quad (10)$$

$$\delta_{\perp}\phi = \delta\phi - \left[\int d^3\vec{x} \delta\phi(\vec{x}) \frac{\partial\phi|_s(\vec{x})}{\partial s} \right] \frac{\partial\phi|_s}{\partial s} \quad (11)$$

where $\delta_{\parallel}\phi$ is the cotangent vector which is parallel to $\frac{\partial\phi|_s}{\partial s}$ and $\delta_{\perp}\phi$ is the cotangent vector which is perpendicular to $\frac{\partial\phi|_s}{\partial s}$ because we can easily prove that $\langle \delta_{\perp}\phi, \frac{\partial\phi|_s}{\partial s} \rangle = 0$. We define the unit normal vector in direction $n(\vec{y})$ to be

$$\hat{\phi}_{n(\vec{y})}(\vec{x}; s) = \frac{\delta_{n(\vec{y})}\phi_{\perp}}{\sqrt{\int d^3\vec{x} (\delta_{n(\vec{y})}\phi_{\perp}(\vec{x}))^2}} \quad (12)$$

so we can define the deviation from the curve which is perpendicular to the tangent vector as

$$\Delta\phi_{\perp}|_s(\vec{x}) = \int d^3\vec{y} \left(n(\vec{y}) \hat{\phi}_{n(\vec{y})}(\vec{x}; s) \right) \quad (13)$$

where $n(\vec{y})$ is the component in the $\vec{y}$ direction. The neighbor of the curve can be described as

$$\phi_{neighbor}(\vec{x}) = \phi(\vec{x}; s) + \Delta\phi_{\perp}(\vec{x}; s) \quad (14)$$

This description of the curve in configuration space will be used in the calculation of MPEP later.

3.2 Wheeler-DeWitt equation in flat spacetime

To derive the Wheeler-DeWitt equation for simple scalar field in flat spacetime, we will use the heuristic method which is the analogy as canonical quantization in QM rather than begin at generic Wheeler-DeWitt equation and take flat spacetime limit [17] [5] [4]. For the simplest scalar field, we can have its Hamiltonian Density $\mathcal{H}(\phi, \dot{\phi}) = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)$. Doing the canonical quantization $\dot{\phi}(\vec{x}) \rightarrow \pi(\vec{x}) = -i\frac{\delta}{\delta\phi(\vec{x})}$, we can have the functional Schrodinger equation

$$H[\phi, \frac{\delta}{\delta\phi}]\Phi[\phi] = E\Phi[\phi] \quad (15)$$

where

$$H[\phi, \frac{\delta}{\delta\phi}] = \int d^3\vec{x} \left[-\frac{1}{2}\hbar^2 \left(\frac{\delta}{\delta\phi} \right)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \right] \quad (16)$$

$\Phi(\phi)$ is the wavefunction in the configuration space and $U(\phi) = \int d^3\vec{x} \left(\frac{1}{2}(\nabla\phi)^2 + V(\phi) \right)$ is the effective potential in the functional Schrodinger equation. We should put an eye on the term $\int d^3\vec{x} \left(\frac{1}{2}(\nabla\phi)^2 \right)$ because this term will modify the profile of $V(\phi)$ significantly [4].

We use the WKB approximation to solve the functional Schrodinger equation. We write the wave functional as $\Phi[\phi] = A \exp(\frac{i}{\hbar}S[\phi])$ where $S[\phi] = S^{(0)}[\phi] + \hbar S^{(1)}[\phi] + \hbar^2 S^{(2)}[\phi] + \dots$ we have that

$$\begin{aligned} & \int d^3\vec{x} \left[-\frac{1}{2}\hbar^2 \left(\frac{\delta}{\delta\phi(\vec{x})} \right)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \right] \exp\left(\frac{i}{\hbar} (S^{(0)}[\phi] + \hbar S^{(1)}[\phi] + \dots) \right) \\ &= E \exp\left(\frac{i}{\hbar} (S^{(0)}[\phi] + \hbar S^{(1)}[\phi] + \dots) \right) \end{aligned} \quad (17)$$

We expand it in terms of $\hbar$ and we can have 0th order equation which is

$$\int d^3\vec{x} \left[\frac{1}{2} \left(\frac{\delta S^{(0)}[\phi]}{\delta\phi(\vec{x})} \right)^2 + \frac{1}{2}(\nabla\phi)^2 + V[\phi] \right] = E \quad (18)$$

and we have 1st order equation which is

$$\int d^3\vec{x} \left[(-i) \frac{\delta^2 S^{(0)}[\phi]}{\delta\phi(\vec{x})^2} + 2 \frac{\delta S^{(0)}[\phi]}{\delta\phi(\vec{x})} \frac{\delta S^{(1)}[\phi]}{\delta\phi(\vec{x})} \right] = 0 \quad (19)$$

These are functional equations which are hard to calculate. To make the equations calculable, we can reduce the functional equation to the calculation on MPEP so we can solve 1D problem rather than infinite dimensional problem.

3.3 Most Probable Escape Path(MPEP)

We can calculate MPEP using the 0th order equation of WKB approximation. The calculation in finite dimensional case has already been done by Tom Banks, Carl M. Bender and Tai Tsun Wu [2] [3], and the calculation in infinitesimal case is done by Khalil M. Bitar, Shau-Jin Chang in 1978 [17]. Now let's follow the calculation by Khalil M. Bitar and Shau-Jin Chang. $S^{(0)}$ can be treated as a field on the configuration space and in 0th order approximation the $S^{(0)}$ gradient $\frac{\delta S^{(0)}[\phi]}{\delta \phi} \Big|_{\phi_0(\lambda)}$ is parallel to the tangent of MPEP $\frac{d\phi_0}{d\lambda}$ and we can have that

$$\frac{\delta S^{(0)}[\phi]}{\delta \phi_{\parallel}} \Big|_{\phi_0(\vec{x}; \lambda)} = C(\lambda) \frac{\partial \phi_0(\vec{x}; \lambda)}{\partial \lambda} \quad (20)$$

$$\frac{\delta S^{(0)}[\phi]}{\delta \phi_{\perp}} \Big|_{\phi_0(\vec{x}; \lambda)} = 0 \quad (21)$$

Doing the analogy with the formula $\frac{\partial S^{(0)}}{\partial \lambda}$ in finite dimension which is $\frac{\partial S^{(0)}}{\partial \lambda} = \nabla S^{(0)} \cdot \frac{d\vec{r}}{d\lambda}$, we have that

$$\frac{\partial S^{(0)}}{\partial \lambda} = \int d^3 \vec{x} \left(\frac{\delta S^{(0)}}{\delta \phi_{\parallel}} \Big|_{\phi_0(\vec{x}; \lambda)} \frac{\partial \phi_0(\vec{x}; \lambda)}{\partial \lambda} \right) \quad (22)$$

Combing equations (20) and (22) we can determine the coefficient $C(\lambda)$ which is

$$C(\lambda) = \frac{\frac{\partial S^{(0)}}{\partial \lambda}}{\int d^3 \vec{x} \left[\frac{\partial \phi_0(\vec{x}; \lambda)}{\partial \lambda} \right]^2} \quad (23)$$

We plug equations (20) and (23) into (18) (19) and choose $\lambda = s$ in the same time, we can reduce the equation in functional space to 1D equations on the trajectory of MPEP which are

$$\frac{1}{2} \left(\frac{\partial S^{(0)}(s)}{\partial s} \right)^2 + U[\phi_0(s)] = E \quad (24)$$

in 0th order expansion and

$$-i \frac{\partial^2 S^{(0)}[\phi]}{\partial s^2} + 2 \frac{\partial S^{(0)}}{\partial s} \frac{\partial S^{(1)}}{\partial s} = 0 \quad (25)$$

in 1st order expansion.

Let's solve 1D 0th order equation (24) and then plug in 1st order equation (25). In classical allowed region, we have that

$$S_A^{(0)}(s) = \int_0^s ds' \sqrt{2(E - U[\phi_0(s')])} \quad (26)$$

$$S_A^{(1)}(s) = \frac{i}{2} \ln(\sqrt{2(E - U[\phi_0(s)])}) \quad (27)$$

and in classical forbidden region, we have that

$$S_F^{(0)}(s) = \int_0^s ds' \sqrt{2(U(\phi_0(s')) - E)} \quad (28)$$

$$S_F^{(1)}(s) = \frac{i}{2} \ln(\sqrt{2(U[\phi_0(s)] - E)}) \quad (29)$$

We determine the MPEP through using Hamilton-Jacobi variation principle on 0th order equation and we can have that

$$\frac{\partial^2 \phi_F(\vec{x}; \lambda)}{\partial \lambda^2} + \nabla^2 \phi_F(\vec{x}; \lambda) - \frac{\partial U[\phi_F(\vec{x}, \lambda)]}{\partial \phi_F} = 0 \quad (30)$$

$$\frac{ds}{d\lambda} = \sqrt{2(U[\phi_0(\lambda)] - E)} \quad (31)$$

and that

$$\frac{\partial^2 \phi_A(\vec{x}; \eta)}{\partial \eta^2} - \nabla^2 \phi_A(\vec{x}; \eta) + \frac{\partial V[\phi_A(\vec{x}; \eta)]}{\partial \phi_A} = 0 \quad (32)$$

$$\frac{ds}{d\eta} = \sqrt{2(E - U[\phi_0(\eta)])} \quad (33)$$

By solving these equations we can calculate MPEP and then solve the functional schrodinger equation in 1D situation. For the calculation in classical forbidden, the solving of MPEP is similar to the solving of bounce action in Coleman's instanton method [9] [10] [11] where we can borrow most of the tricks from Coleman's instanton to the calculation of MPEP.

4 Resonant Tunneling

In non-relativistic quantum mechanics, we can use the WKB approximation. Here to be coincide with the treatment of [6] [4] [7] [5], let's use the formalism of [1].

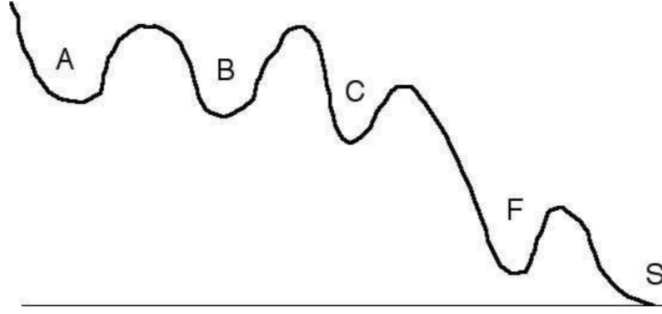


Figure 2: String theory shows that there are exponentially or even infinitely many metastable solutions which makes resonant tunneling easy to happen. Here we give a schematic picture of the topography of the potential. Here A B and C are three metastable vacuums and B is in the middle of A and C. What we are discussing is the tunneling from A to C through B. In functional schrodinger formalism, the contribution from classical allowed region B can provide important phase factor which makes the resonance tunneling possible to happen. However, the condition of resonant tunneling to happen is strict which must satisfy $W = (n + \frac{1}{2})\pi$. The width of resonant spectrum is exponentially suppressed.

In classical allowed region, we have left-moving and right-moving wave functions which are

$$\Psi_{L,R}(x) \approx \frac{1}{\sqrt{k(x)}} \exp\left(\pm i \int^x dx k(x)\right) \quad (34)$$

where

$$k(x) = \sqrt{\frac{2m(E - V(x))}{\hbar^2}} \quad (35)$$

So the whole wave function is

$$\Psi(x) = \alpha_R \Psi_R(x) + \alpha_L \Psi_L(x) \quad (36)$$

In classical forbidden region, we have exponentially decay and exponentially growth wave functions

$$\Psi_{\pm} \approx \frac{1}{\sqrt{\kappa(x)}} \exp\left(\pm \int^x dx \kappa(x)\right) \quad (37)$$

where

$$\kappa(x) = \sqrt{\frac{2m(V(x) - E)}{\hbar^2}} \quad (38)$$

So the whole wave function is

$$\Psi(x) = \alpha_+ \Psi_+(x) + \alpha_- \Psi_-(x) \quad (39)$$

In single-barrier case, when tunneling happens from vacuum A to vacuum B, we have that

$$\begin{bmatrix} \alpha_R \\ \alpha_L \end{bmatrix} = \left(\frac{1}{2}\right)^2 \begin{bmatrix} \Theta + \frac{1}{\Theta} & i(\Theta - \frac{1}{\Theta}) \\ -i(\Theta - \frac{1}{\Theta}) & \Theta + \frac{1}{\Theta} \end{bmatrix} \begin{bmatrix} \beta_R \\ \beta_L \end{bmatrix} \quad (40)$$

In multi-barrier case, when tunneling happens from vacuum A to vacuum C (Vacuum B is in the middle), we have that

$$\begin{bmatrix} \alpha_R \\ \alpha_L \end{bmatrix} = \left(\frac{1}{2}\right)^2 \begin{bmatrix} \Theta + \frac{1}{\Theta} & i(\Theta - \frac{1}{\Theta}) \\ -i(\Theta - \frac{1}{\Theta}) & \Theta + \frac{1}{\Theta} \end{bmatrix} \begin{bmatrix} e^{-\frac{i}{\hbar}W} & 0 \\ 0 & e^{+\frac{i}{\hbar}W} \end{bmatrix} \begin{bmatrix} \Phi + \frac{1}{\Phi} & i(\Phi - \frac{1}{\Phi}) \\ -i(\Phi - \frac{1}{\Phi}) & \Phi + \frac{1}{\Phi} \end{bmatrix} \begin{bmatrix} \gamma_R \\ \gamma_L \end{bmatrix} \quad (41)$$

In QM,

$$\Theta \approx 2 \exp\left(\frac{1}{\hbar} \int_{x_1}^{x_2} dx \sqrt{2m(V(x) - E)}\right) \quad (42)$$

$$\Phi \approx 2 \exp\left(\frac{1}{\hbar} \int_{x_3}^{x_4} dx \sqrt{2m(V(x) - E)}\right) \quad (43)$$

$$W = \int_{x_2}^{x_3} dx \sqrt{2m(E - V(x))} \quad (44)$$

In scalar QFT, since all the calculations are reduced to 1D problem parameterized by λ , we can also have that

$$\Theta \approx 2 \exp\left(\frac{1}{\hbar} \left[\int_{\lambda_1}^{\lambda_2} d\lambda' \sqrt{2m(\lambda') [U_{eff}(\lambda') - E]} \right]\right) \quad (45)$$

$$\Phi \approx 2 \exp\left(\frac{1}{\hbar} \left[\int_{\lambda_3}^{\lambda_4} d\lambda' \sqrt{2m(\lambda') [U_{eff}(\lambda') - E]} \right]\right) \quad (46)$$

$$W = \int_{\lambda_2}^{\lambda_3} d\lambda' \sqrt{2m(E - U_{eff}(\lambda'))} \quad (47)$$

The effective potential is that $U_{eff}(\lambda) = U(\lambda) + U^{1-loop}(\lambda)$, where $U^{1-loop}(\lambda)$ is 1-loop correction when integrating out the fluctuations denoted by δn . The 1-loop correction term is evaluated in this way

$$\exp\left(-\frac{1}{\hbar} \int_{\tau_1}^{\tau} F(\tau') d\tau'\right) = \frac{1}{\sqrt{\det' \left(-\frac{\partial^2}{\partial \tau^2} \nabla^2 + \frac{\partial^2 V(\phi)}{\partial \phi^2} \right)}} \quad (48)$$

After some algebra, we can derive the tunneling probability from vacuum A to vacuum B, which is

$$T_{A \rightarrow B} = \left| \frac{\beta_R}{\alpha_R} \right|^2 = 4 \left(\Theta + \frac{1}{\Theta} \right)^{-2} \approx \frac{4}{\Theta^2} \quad (49)$$

In the single barrier case, the tunneling probability is exponentially suppressed. We may naively assume that in double barrier case, the tunneling probability is double exponentially suppressed, however, we can find that because of phase factor contribution from classical allowed region, the resonant tunneling will happen and the tunneling probability can be exponentially enhanced in some particular region of parameter space.

We can also derive the tunneling probability from vacuum A to vacuum C(Through B), which is

$$T_{A \rightarrow B \rightarrow C} = \left| \frac{\gamma_R}{\alpha_R} \right|^2 = \frac{4}{(\Theta\Phi + \frac{1}{\Theta\Phi})^2 \cos^2(W) + (\frac{\Theta}{\Phi} + \frac{\Phi}{\Theta})^2 \sin^2(W)} \quad (50)$$

If the vacuum B has zero width which means $W = 0$, we can have that

$$T_{A \rightarrow B \rightarrow C} = \frac{4}{(\Theta\Phi + \frac{1}{\Theta\Phi})^2} \approx \frac{4}{(\Theta\Phi)^2} = \frac{T_{A \rightarrow B} T_{B \rightarrow C}}{4} \quad (51)$$

this means the tunneling probability from A to C is double exponentially suppressed. When $W = (n + \frac{1}{2})\pi$, the tunneling probability is

$$T_{A \rightarrow B \rightarrow C} = \frac{4}{(\frac{\Theta}{\Phi} + \frac{\Phi}{\Theta})^2} \quad (52)$$

the tunneling probability is not exponentially suppressed and the resonant tunneling happens. When $W = (n + \frac{1}{2})\pi$ and $\Theta = \Phi$, the tunneling rate is $T_{A \rightarrow B \rightarrow C} = 1$.

5 Conclusion

In this article, firstly we introduced Coleman's instanton method not only in zero temperature situation but also in finite temperature situation. Secondly, we derived Wheeler-DeWitt equation using a heuristic way. Then we give the method of calculating MPEP and show how to reduce the whole calculation in infinite dimensional spacetime to 1D space. Finally, we showed the condition of resonant tunneling to happen in QM and scalar QFT. The resonant tunneling will find its important application in the constraining of parameter space in some low energy effective theory of string theory on inflation(which have multi-vacua).

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