Understanding WZW Theories with Chern-Simons Theories and Vice-Versa

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Abstract

We examine the correspondence between 2+1 dimensional pure Chern-Simons gauge theories with a semi simple, compact gauge group and 1+1 dimensional Wess-Zumino-Witten (WZW) conformal field theories. We first introduce WZW models as conformal field theories with an additional symmetry group described by an algebra of current operators. The operator product expansions and the stress tensor of the theory are calculated and compared to those of a conventional CFT. We then introduce the classical Chern-Simons action, quantize it by imposing the Gauss law constraint, and show that the physical wave functionals satisfying the constraint correspond exactly to the path integral of the WZW action with the Chern-Simons gauge field A as a source. This correspondence is the core of the relationship between CS and WZW theories. We go on to compute WZW current correlators as well as Chern-Simons Wilson lines. The latter are shown to correspond to primary field insertions in the WZW generating functional.

1 Introduction

Since the mid 1990's, the Holographic Principle has been driving much of the inquiry in theoretical physics. It's realization in the AdS/CFT conjecture allows us to relate strongly coupled conformal field theories in the large-N limit to theories of semiclassical gravity in one dimension higher. In this paper we will explore a predecessor to the modern picture of holography. Namely, we will see that Chern-Simons theories quantized on closed manifolds can be described exactly by the path integral of a WZW theory in one less dimension. Though this is an exact correspondence, whereas AdS/CFT requires c, the central charge, of the boundary CFT to be large, it is in a similar spirit, and could be used to motivate further study in holography.

2 Wess-Zumino-Witten Theories

Suppose we want to write down a conformal field theory with a global symmetry under some Lie Group G (we will restrict ourselves to compact, semi simple groups in this paper). Classically, this means our action should be a scalar under G. We have already seen (e.g. in Polchinski ch. 2) how a nonlinear sigma model of a noncompact scalar boson X^μ has a manifest symmetry under translations of the field values, $X^\mu \to X^\mu + \delta X^\mu$.

In searching for higher symmetry, we could naively extend the NLSM to a bosonic field taking values in a unitary representation of our group G. Call this field $g(z)$. Our first pass at writing a G-invariant action is thus

$$S_0 = \frac{1}{4\lambda^2} \int_{\mathcal{M}} d^2x Tr(\partial_\mu g \partial^\mu g^{-1})$$

where the trace is over the matrix indices in the algebra. $\mathcal{M}$ is a compact 2D manifold.

The action S_0 is indeed conformally invariant classically (λ is dimensionless). However, it is possible to add marginally relevant perturbations to the theory and get a nonzero β-function out the other side, indicating that the scale-invariance is broken quantum mechanically.

On top of the non-trivial RG flow, S_0 fails to produce separately conserved Kac-Moody currents. Our conserved current is $J_\mu = g^{-1}\partial_\mu g$, and one can easily check that for a nonabelian algebra, the holomorphic and anti holomorphic part of the current are only conserved together, not independently. It turns out that currents of the following form

$$J_z = \partial_z g g^{-1}$$
$$J_{\bar{z}} = g^{-1}\partial_{\bar{z}} g$$

do have the desired property. We will thus augment our action by another term that will initially seem arbitrary, but will make sense in hindsight. The new term is the Wess-Zumino (WZ) term (Witten gets credit for adding it to the NLSM action). Our total action is now

$$S = S_0 + k\Gamma$$

where

$$\Gamma[g] = \frac{i}{24\pi} \int_{\mathcal{B}} d^3x \epsilon_{\mu\nu\eta} Tr(g^{-1}\partial^\mu g g^{-1}\partial^\nu g g^{-1}\partial^\eta g)$$

where $\mathcal{B}$ is a 3D manifold with $\mathcal{M}$ as its boundary and k is a constant that will be fixed shortly. It seems odd that we are adding integrals over manifolds of different dimension, but we can make sense of this in two ways. First, we can observe that under the transformation $g \to g + \delta g$, the WZ term's variation is a total derivative, and can thus be written as an integral over $\mathcal{M}$

$$\delta\Gamma = \frac{i}{8\pi}\int_{\mathcal{M}} d^2x\epsilon_{\mu\nu}Tr(g^{-1}\delta g\partial^{\mu}(g^{-1}\partial^{\nu}g))$$

Mathematically, the WZ term tells us about the third homotopy class, $\pi_3(G)$, of the map $g : \mathcal{B} \to G$. Thus it is always an integer (since we have restricted ourselves to compact, semi simple groups, for which $\pi_3(G) = \mathbb{Z}$). For our path integral to be gauge invariant, k must therefore be quantized as well.

One important identity that the full WZW action satisfies is the Polyakov-Wiegmann identity

$$S[gh^{-1}] = S[g] + S[h^{-1}] + \frac{k}{2\pi}\int d^2zTr(g^{-1}\partial_{\bar{z}}gh^{-1}\partial_z h)$$

Which means that our action transforms as a 1-cocycle, indicating the presence of a projective representation of our gauge group. This is the basic root of why the WZW action produces the Kac-Moody current algebra.

The upshot of adding this new term can be seen from the new equation of motion:

$$\partial^{\mu}(g^{-1}\partial_{\mu}g) + \frac{ik\lambda^2}{4\pi}\epsilon_{\mu\nu}\partial^{\mu}(g^{-1}\partial_{\nu}g) = 0$$

which becomes the following in complex coordinates:

$$(1 + \frac{k\lambda^2}{4\pi})\partial_z(g^{-1}\partial_{\bar{z}}g) + (1 - \frac{k\lambda^2}{4\pi})\partial_{\bar{z}}(g^{-1}\partial_z g) = 0$$

For $k = \frac{4\pi}{\lambda^2}$ and $k = -\frac{4\pi}{\lambda^2}$ we have two separate conservation laws for two separate currents. It appears we have found our Kac-Moody currents! Indeed, the two solutions correspond to IR fixed points of the NLSM, and flowing away from them yields asymptotically free theories. Normalizing the currents, we have

$$J(z) = -k\partial_z gg^{-1}$$

$$\bar{J}(\bar{z}) = kg^{-1}\partial_{\bar{z}}g$$

Written as a sum of Lie algebra generators T^a the holomorphic part of the current is

$$J = \sum_a J^a T^a$$

Using this to apply an infinitesimal gauge transformation, $g \to g + Ag - g\bar{A}$ (where A is some element of the Lie algebra) to some operator X, we get the following Ward identity

$$\delta\langle X\rangle = \frac{i}{2\pi}\oint dz A^a \langle J^a X\rangle - \frac{i}{2\pi}\oint d\bar{z}\bar{A}^a\langle \bar{J}^a X\rangle$$

The transformation properties of the current itself can be found directly from its definition:

$$\delta J = [A, J] - k\partial_z A$$

Substituting J^b for the operator X in the Ward identity, and combining it with the above expression yields the following OPE

$$J^a(z)J^b(w) = \frac{k\delta_{ab}}{(z-w)^2} + if_{abc}\frac{J^c(w)}{(z-w)} + \ldots$$

where f_{abc} are the structure constants of the Lie algebra g. The leading order term has a constant coefficient, and tells us that the currents form a projective representation of the Lie algebra of LG, the loop group associated with G. Equivalently, we can say that they form a non-projective representation of the central extension of the Lie algebra of LG, $\hat{LG}$. This is very familiar from our quantization of conventional CFTs, where the Virasoro algebra turns out to be a central extension of the classical conformal (Witt) algebra. To make the the connection with the Virasoro algebra clearer, we can perform a mode expansion of the current operators

$$J^a(z) = \sum_n z^{-n-1}J^a_n$$

Combining this and the OPE, we arrive at the current algebra

$$[J^a_n, J^b_m] = if_{abc}J^c_{n+m} + kn\delta_{ab}\delta_{n+m,0}$$

Thus we see that the central extension of our algebra g has a similar structure as the Virasoro algebra when expressed in terms of modes. This infinite-dimensional algebra is the affine Lie algebra $\hat{g}_k$ at level k. We see that it reduces to the original, finite-dimensional Lie algebra g if $n = m = 0$. Turning global G-invariance into local $G(z)$-invariance (which is what is expressed by the mode expansion of J), and taking into account quantum effects arising from normal ordering and anomalies, we arrive at a much richer symmetry algebra than we started with in the classical action.

2.1 The Sugawara Construction

One possible lurking issue is that we have not fully demonstrated that this theory remains conformal after quantization (besides stating without proof that it is an IR fixed point). If our theory is to have local conformal invariance as do the other 2D CFTs we have encountered, the modes of the quantum mechanical stress energy tensor must form a representation of the Virasoro algebra. This turns out to be exactly the case. The Sugawara construction allows us to build a stress tensor out of a sum of current bilinears.

The classical stress tensor can be derived from the classical WZW action by varying with respect to the background metric, and has the following form:

$$T_{cl}(z) = \frac{1}{2k}\sum_a (J^a J^a)(z)$$

Let's assume that we can indeed build a stress tensor with a similar form for the quantum theory. We will start with

$$T(z) = \gamma \sum_a (J^a J^a)(z)$$

where normal ordering is implied and we will fix γ by finding the TT OPE and requiring it to have the standard form. γ will thus take quantum effects fully into account. Using the JJ OPE as well as the following relation

$$\sum_{b,c} f_{abc} f_{dbc} = 2g\delta_{ad}$$

where g is the dual Coxeter number of the group, we arrive at

$$\gamma = \frac{1}{2(k+g)}$$

This also tells us that our central charge is

$$c = \frac{kdim(g)}{k+g}$$

The intermediate TJ OPE also tells us that J is a Virasoro primary field with dimension 1. The remarkable thing about this Sugawara construction is that, starting with our affine current algebra, we were able to construct Virasoro algebra generators by taking bilinear combinations of currents. Intuitively we should think that the Virasoro algebra is somehow smaller than the current algebra. More specifically, it turns out that the Virasoro algebra is actually part of the universal enveloping algebra of $\hat{g}_k$. The full algebra $Vir \ltimes \hat{g}_k$ can be summarized by the following Lie bracket relations:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}$$
$$[L_n, J^a_m] = -mJ^a_{n+m}$$
$$[J^a_n, J^b_m] = if_{abc}J^c_{n+m} + kn\delta_{ab}\delta_{n+m,0}$$

It should be noted that $\hat{g}_k$ is not completely a symmetry of the theory, but it can be used to generate the full Hilbert space of physical states, in the same way we generate states with the Virasoro algebra.

3 Moving to 3D

We will now review the quantization of a Chern-Simons theory on a compact 2D spatial manifold. Our space-time manifold will have the form $\mathcal{M} \times \mathbb{R}$ where, again, $\mathcal{M}$ is a compact 2D manifold (e.g. the 2-torus), and $\mathbb{R}$ parameterizes time. A is a 3-component g-valued gauge field in a unitary representation of the Lie algebra, g. We can write this action in a completely coordinate independent way with differential forms:

$$S_{CS} = \frac{k}{4\pi} \int_{\mathcal{M}} Tr(AdA + \frac{2}{3}A^3)$$

Where all products are wedge products. Here the metric independence of the theory is manifest. The sourceless stress energy tensor therefore vanishes, telling us that our theory is completely invariant under space-time transformations and can therefore be called topological, i.e. independent of local geometry.

Let us look into the gauge structure of the classical action. Under a gauge transformation $A_\mu \to g^{-1}A_\mu g + g^{-1}\partial_\mu g$, our action changes as follows:

$$\delta S = \frac{k}{4\pi} \int_{d\mathcal{M}} Tr(\delta AA) + \frac{k}{2\pi} \int_{\mathcal{M}} Tr(\delta AF)$$

The first term is purely a boundary term, which does not matter for us. The second term is a constraint on the classical field values. It tells us that although our theory does not have a Hamiltonian, the symplectic structure of the classical theory is not totally trivial.

The second term becomes

$$\frac{1}{24\pi^2} Tr(g^{-1}dg)^3$$

which is exactly the Wess-Zumino term, indicating that the Chern-Simons action is sensitive to large gauge transformations in a similar way to the WZW action.

Thus, the action of our Chern-Simons gauge theory is *not* gauge-invariant. However, since the change induced in the action by a gauge transformation is always integral, our partition function remains gauge-invariant *if* k, the level, is an integer. This term thus tells us about, $\pi_3(G)$, the third homotopy group of our Lie group manifold. Since we are restricting ourselves to a compact groups, it will always be the case that $\pi_3(G) = \mathbb{Z}$. This will turn out to be the same k as in the WZW action. The 1-cocycle transformation property of the WZW action will end up being required of the physical wave functions of the Chern-Simons theory.

The classical equations of motion for a sourceless Chern-Simons theory in coordinate-free notation are

$$dA + A^2 = F = 0$$

which are trivially solved by pure gauge, flat connections, $A_\mu = g^{-1}\partial_\mu g$.

3.1 Quantizing Chern-Simons Theory

Since the Chern-Simons action is first order in time derivatives, it has no Hamiltonian. Naively we might be temped to say the quantum mechanical theory is thus trivial. However, in quantizing the theory, we inherit both the canonical form of the phase space manifold:

$$\omega = -\frac{k}{4\pi}\int Tr(\delta A \delta A)$$

as well as a constraint given by recognizing that A_0 is simply a Lagrange multiplier. Decomposing the field into space and time components, $A = \tilde{A} + A_0$, makes the constraint obvious:

$$S = -\frac{k}{4\pi}\int Tr(\tilde{A}\partial_t \tilde{A}) + \frac{k}{2\pi}\int Tr(A_0(d\tilde{A} + \tilde{A}^2))$$

Thus, we must quantize the Poisson bracket of the theory, fix the gauge (since A_0 is completely redundant), and impose the following constraint:

$$d\tilde{A} + \tilde{A}^2 = 0$$

This constraint is simply the spatial equation on motion, known as the Gauss-law constraint for obvious reasons. This constraint restricts us classically to the space of flat (pure gauge) connections. However, we must be careful when imposing this constraint. It turns out that because of operator-ordering ambiguity, it matters whether we impose the constraint and then quantize the restricted phase space, or we quantize the full space and *then* impose the constraint. We we follow the second procedure so as to avoid the ordering ambiguity.

Let us begin the quantization procedure. Referring back to our action decomposed into time and space components, we can pick the axial gauge, $A_0 = 0$, for the Lagrange multiplier A_0. However, this does not mean we can ignore the Gauss law constraint that A_0 enforces in the action. We will implement that later once we construct our canonical operators. After fixing the gauge, we have the following action:

$$\frac{k}{8\pi}\int_{\mathcal{M}} \epsilon^{ij}\dot{A}_i^a A_j^a$$

where we have used the orthogonality of the trace on the Lie algebra basis, $Tr(T^a T^b) = \frac{\delta^{ab}}{2}$, to write the action in terms of the coefficients of the generators. We can extract the Poisson bracket of this theory quite easily:

$$\{A_i^a(x), A_j^b(y)\} = \epsilon_{ij}\delta^{ab}\delta^2(x-y)$$

promoting this to a Dirac bracket and normalizing by k we get

$$[A_i^a(x), A_j^b(y)] = \frac{2\pi i}{k}\epsilon_{ij}\delta^{ab}\delta^2(x-y)$$

where the components of A are now operators, of course. We will transform to complex coordinates, $z, \bar{z}$, and one can verify that the commutator remains essentially the same:

$$[A_z^a(z_1), A_{\bar{z}}^b(z_2)] = \frac{2\pi i}{k}\delta^{ab}\delta^2(z_1 - z_2)$$

To obtain a Schrodinger representation of these operators, we will arbitrarily pick A_1 to be our coordinate and A_2 to be the conjugate momentum. That is, A_2 will act on position space wave functionals by functional differentiation by A_1.

$$A_{\bar{z}}^a \Psi(A_z^b) = -\frac{i2\pi}{k}\frac{\delta}{\delta A_z^a(x)}\Psi(A_z^b)$$

We can easily verify that this representation of the operators obeys the commutation relation.

Now our field strength becomes (temporarily suppressing gauge bundle indices)

$$F_{z,\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}]$$

This generates the gauge transformations, $U(g) = e^{i\int \omega F}$, where ω is a vector of parameters that exponentiate the Lie algebra to the group element g. We can check that F does indeed induce a representation of the group via

$$U(g_1)U(g_2) = U(g_1 g_2)$$

The full transformation $U(g)$ must therefore act as the identity on physical states, meaning that F must annihilate them:

$$F|\psi\rangle = 0$$

In our position space representation, this condition simply gives us a functional differential equation:

$$\Big(\partial\frac{\delta}{\delta A_z} + \big[A_z, \frac{\delta}{\delta A_z}\big] - \frac{ik}{2\pi}\bar{\partial}A_z\Big)\Psi(A_z) = 0$$

Though this equation might seem somewhat opaque, a bit of manipulation will bring it into a familiar form. First, we assume that up to normalization, the wave function has the form $\Psi[A_z] = exp(iW[A_z])$, where W is a functional of A. Then we bring back our group indices and use our canonical commutation relations to get:

$$(\delta^{ac}\partial + f^{abc}A_z^b)\frac{\delta W(A_z)}{\delta A_z^c}\Psi = -\frac{k}{4\pi}\partial A^a \Psi$$

Comparing this result to the previous section, this turns out to be *exactly* the form of the differential equation satisfied by the generating functional for Kac-Moody current correlators.

$$\Psi[A_z] = \int Dg e^{iS_{WZW} + ik\int_{\mathcal{M}} \frac{d^2z}{4\pi} Tr A_z g^{-1}\bar{\partial} g}$$

or

$$\boxed{\Psi[A_z] = \int Dg e^{iS_{WZW} + i\int_{\mathcal{M}} \frac{d^2z}{4\pi} A_z^a \bar{J}^a}}$$

4 WZW-CS Correspondence

Reiterating the result of the previous section, we found that physical Chern-Simons wave functionals quantized on an arbitrary closed $2D$ manifold are identical to the WZW path integral generating functionals. This means that by taking functional derivatives of $\Psi[A]$ with respect to A we can calculate arbitrary n-point functions of the current operators. For example, we can calculate the 2-point current correlator as follows:

$$\langle \bar{J}^a(z)\bar{J}^b(w)\rangle \sim \frac{\delta}{\delta A^a(z)}\frac{\delta}{\delta A^b(w)}\Psi[A_z] \sim \frac{k\delta^{ab}}{(z-w)^2}$$

This agrees with our OPE (by design). Next we can calculate something more complicated, like the 3-point function:

$$\langle \bar{J}^a(z)\bar{J}^b(w)\bar{J}^c(x)\rangle \sim \frac{\delta}{\delta A^a(z)}\frac{\delta}{\delta A^b(w)}\frac{\delta}{\delta A^c(x)}\Psi[A_z] \sim \frac{ikf^{abc}}{(z-w)(z-y)(w-y)}$$

One can verify that this agrees with the original OPE. We can easily take this to higher order, but the correlation functions rapidly become very complicated. They all, however, retain the expected transformation properties under conformal transformations.

4.1 Adding Sources

Finally, we can see what happens if we add sources to our Chern-Simons Lagrangian. The source term has the form

$$i\int dt \sum{}_{\alpha} T^a_\alpha A^a_0$$

where the Ts are the Lie algebra elements. This corresponds to inserting stationary charges in the Chern-Simons theory. They can also be thought of as timelike Wilson lines. Each charge transforms under some representation of G, and the resulting Hilbert space becomes the direct product of the space of sourceless wave functionals with the representation spaces of all of the N inserted charges. We can see this by noting that the Gauss law constraint is altered to be

$$\frac{k}{4\pi}F(z)\Psi = \sum_{\alpha}^{N} \rho_\alpha(T^a)\delta^2(z - z_\alpha)\Psi$$

where $\rho_\alpha(T^a)$ is the matrix corresponding to the generator T^a in the representation under which charge α transforms. Reevaluating our functional differential equation for $\Psi[A]$, we end up with

$$\Psi[A; z_1, z_2, ..., z_N] = \int Dg\phi_1(z_1)\phi_2(z_2)...\phi_N(z_N)e^{iS_{WZW} + i\int_{\mathcal{M}} \frac{d^2z}{4\pi} A_z^a \bar{J}^a}$$

Here $\phi_\alpha(z_\alpha) = e^{-i\int dt \rho_\alpha(T^a_\alpha)A^a_0}$. i.e. each ϕ is an operator transforming under a particular representation of the gauge group. Our sourced Chern-Simons wave functional corresponds to the expectation value of the product of these operators in the dual WZW model. However, the fields transforming under the representations of the gauge group correspond precisely to the Kac-Moody primary fields (and thus the Virasoro primary fields) of the WZW model. Thus, there is a direct correspondence between primary fields in the WZW model and the Wilson lines in the Chern-Simons theory. Since the Wilson operators are the only gauge-invariant observables in the Chern-Simons theory, it follows that we can calculate the expectation values of any physical operator in Chern-Simons theory by calculating the expectation value of primary fields in a WZW model.

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