Grading.

The maximum score on this problem set was $10\,\frac{\text{points}}{\text{problem}} \cdot 6$ problems $= 60$ points.

Email cferko@uchicago.edu with questions or corrections.

Exercise 1. 10 points

Take the action for a point particle with dynamical einbein $e(\tau)$ and $X(\tau)$ fields:

$$S = \frac{1}{2} \int d\tau \left(e^{-1} \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu} - e m^2 \right) . \tag{1}$$

Find the equations of motion. Integrate out e and recover the relativistic point particle action discussed in lecture.

Solution 1.

For each independent field $\phi(\tau)$ on the worldline, the Euler-Lagrange equation is

$$\frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi}, \tag{2}$$

where an overdot represents differentiation with respect to τ.

First apply equation (2) to the einbein $\phi(\tau) = e(\tau)$, which gives

$$\frac{\partial L}{\partial e(\tau)} = - \frac{\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}}{e(\tau)^2} - m^2 = 0, \tag{3}$$

since the Lagrangian is independent of $\dot{e}(\tau)$.

Likewise, when we let $\phi(\tau) = X^\alpha(\tau)$, the Euler-Lagrange equation (2) yields

$$\frac{d}{d\tau} \left(e^{-1} \dot{X}_\alpha \right) = 0, \tag{4}$$

where $\dot{X}_\alpha = \dot{X}^\beta \eta_{\alpha\beta}$.

Next, we "integrate out" the field $e(\tau)$, which means to solve its equation of motion and then substitute back into the action. Solving equation (3) gives

$$e(\tau) = \sqrt{ - \frac{\dot{X}^\mu \dot{X}_\mu}{m^2} }, \tag{5}$$

where we take the positive square root since, morally, $e(\tau) = \sqrt{-g_{\tau\tau}}$ is a frame field on the worldline and should preserve orientation.

Now (5) can be substituted back into (1) to find

$$\begin{aligned} S &= \frac{1}{2} \int d\tau \left(\dot{X}^\mu \dot{X}_\mu \left(- \frac{\dot{X}^\nu \dot{X}_\nu}{m^2} \right)^{-1/2} - m^2 \left(- \frac{\dot{X}^\nu \dot{X}_\nu}{m^2} \right)^{1/2} \right) \\ &= \frac{1}{2} \int d\tau \left(-m \sqrt{ - \dot{X}^\mu \dot{X}_\mu } - m \sqrt{ - \dot{X}^\mu \dot{X}_\mu } \right) \\ &= -m \int d\tau \sqrt{ - \dot{X}^\mu \dot{X}_\mu } \\ &= S_{\text{pp}} . \end{aligned} \tag{6}$$

We see that the einbein point particle action is classically equivalent to the ordinary point particle action which measures the invariant length of the worldline.

Exercise 2. *Polchinski 1.1* 10 points

(a) Show that in the nonrelativistic limit the action S_{pp} has the usual nonrelativistic form, kinetic energy minus potential energy, with the potential energy being the rest mass.

(b) Show that for a string moving nonrelativistically, the Nambu-Goto action reduces to a kinetic term minus a potential term proportional to the length of the string. Show that the kinetic energy comes only from the transverse velocity of the string. Calculate the mass per unit length, as determined from the potential term and also from the kinetic term.

Solution 2.

(a) Painful though it may be, we should now re-introduce factors of c.

Let the position of our particle be $X^\mu(\tau) = \big(ct(\tau), x^1(\tau), \cdots, x^{D-1}(\tau)\big)$ so

$$\dot{X}^\mu(\tau) = \left(c\frac{dt}{d\tau}, \frac{dx^1}{d\tau}, \cdots, \frac{dx^{D-1}}{d\tau}\right). \tag{7}$$

We must make a gauge choice for the worldline parameter τ. For instance, we might choose static gauge $\tau = t$. Alternatively, to make contact with the usual treatment of special relativity, we could let τ be proper time, so the quantity $\frac{dt}{d\tau}$ is $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$. With this choice, in the non-relativistic limit, $\frac{dt}{d\tau} = \gamma = 1 + \mathcal{O}\left(\frac{v^2}{c^2}\right)$, so this is actually equivalent to static gauge to required order in $\frac{v}{c}$.

With either gauge choice for τ, we find

$$\dot{X}^\mu \dot{X}_\mu = -c^2 + |\vec{v}|^2 + \mathcal{O}\left(\frac{v^2}{c^2}\right), \tag{8}$$

where $\vec{v}$ is the spatial vector $\left(\frac{dx^1}{dt}, \cdots, \frac{dx^{D-1}}{dt}\right)$ with norm $|\vec{v}| \equiv v$.

The point particle action, to required order in $\frac{v}{c}$, becomes

$$\begin{aligned} S_{\text{pp}} &\approx -mc \int dt\, \sqrt{c^2 - |\vec{v}|^2} \\ &\approx -mc^2 \int dt\, \left(1 - \frac{1}{2}\frac{|\vec{v}|^2}{c^2}\right), \end{aligned} \tag{9}$$

where we have applied the Taylor expansion

$$(1 + \smiley)^n = 1 + n\smiley + \mathcal{O}\left(\smiley^2\right) \tag{10}$$

to the small quantity $\smiley = \frac{v}{c}$.

We then see that

$$S_{\text{pp}} \approx \int dt\, \left(\frac{1}{2}m\,|\vec{v}|^2 - mc^2\right). \tag{11}$$

This is the time integral of the classical kinetic term, $\frac{1}{2}m\,|\vec{v}|^2$, minus a "potential" term mc^2 arising from the relativistic rest energy.

Calling this a "potential" is a bit misleading, though, since it is a total time derivative which doesn't affect the equations of motion.

(b) The Nambu-Goto action is

$$S_{\text{NG}} = -\frac{1}{2\pi\alpha'}\int d^2\sigma\, \sqrt{-\det h_{ab}}, \tag{12}$$

where h_{ab} is the pullback of the flat Minkowski metric in spacetime, defined by

$$h_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}. \tag{13}$$

As in part (a), we will re-introduce factors of c, writing the embedding coordinates $X^\mu = (ct, x^1, \cdots, x^{D-1})$. Then the elements of h_{ab} are

$$\begin{aligned} h_{\tau\tau} &= \dot{X}^\mu \dot{X}_\mu = -c^2 \left(\frac{dt}{d\tau}\right)^2 + |\vec{v}|^2\,, \\ h_{\sigma\tau} = h_{\tau\sigma} &= \dot{X}^\mu X'_\mu = -c\frac{dt}{d\tau}\frac{dt}{d\sigma} + \vec{v}\cdot\frac{d\vec{x}}{d\sigma}, \\ h_{\sigma\sigma} &= X'^\mu X'_\mu = -c^2\left(\frac{dt}{d\sigma}\right)^2 + \left|\frac{d\vec{x}}{d\sigma}\right|^2. \end{aligned} \tag{14}$$

As usual, we have defined $\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}$, and $X'^\mu = \frac{\partial X^\mu}{\partial \sigma}$, and written the quantities $\vec{v}$ and $\vec{x}$ to denote the spatial vector parts of $\dot{X}^\mu$ and X^μ, respectively.

Now we will express the quantity $\sqrt{-\det h_{ab}}$ appearing in the Nambu-Goto action in terms of the matrix elements in (14). This gives

$$\begin{aligned} \sqrt{-\det h_{ab}} &= \sqrt{-\left(h_{\tau\tau}h_{\sigma\sigma} - h_{\tau\sigma}^2\right)} \\ &= \sqrt{\left(-c\frac{dt}{d\tau}\frac{dt}{d\sigma} + \vec{v}\cdot\frac{d\vec{x}}{d\sigma}\right)^2 - \left(-c^2\left(\frac{dt}{d\tau}\right)^2 + |\vec{v}|^2\right)\left(-c^2\left(\frac{dt}{d\sigma}\right)^2 + \left|\frac{d\vec{x}}{d\sigma}\right|^2\right)}. \end{aligned} \tag{15}$$

To proceed, we must make a gauge choice as in part (a) – that is, we should spend our reparameterization invariance to simplify (15) and compare it to the classical stretched string.

First, let's make the usual static gauge choice for time: $X^0 = c\tau = ct$, or $t = \tau$. (I am setting $R = 1$ in the form of static gauge used by Polchinski, $t = R\tau$.)

We are free to choose one of our spatial coordinates, say $X^1 = x^1 \equiv x$, to parameterize the longitudinal direction along the string. This is allowed for any motion in which X^1 is an increasing function along the string, but in the non-relativistic limit with small oscillations, we may always rotate coordinates so that this is true. Hence we will take

$$X^1 = x = \frac{\ell\sigma}{\pi}, \tag{16}$$

which has been normalized for a string of length ℓ and $\sigma \in [0, \pi]$, appropriate for an open string.

The other embedding functions $X^2, \cdots, X^{24}$ therefore represent the coordinates transverse to the string, and are functions of $X^0 = t = \tau$ and $X^1 = x = \frac{\ell\sigma}{\pi}$. To emphasize that they are transverse, I will re-label these coordinates as Y^i and use the vector notation $\vec{Y} \equiv \left(X^2, \cdots, X^{24}\right) \equiv \left(Y^2, \cdots, Y^{24}\right)$.

Finally, we take the non-relativistic limit, which amounts to

$$\left|\frac{\partial \vec{Y}}{\partial x}\right| \ll 1, \text{ and} \tag{17}$$

$$\left|\frac{1}{c}\frac{\partial \vec{Y}}{\partial t}\right| \ll 1. \tag{18}$$

The assumption (18) is simply that the transverse motion of the string is much slower than the speed of light. The assumption (17) is that the *spatial* gradient of the stretched string is small; one can think of it as a small-amplitude limit obtained via our choice of parameterization.

Using these assumptions, and our gauge choice, the expression (15) becomes

$$\sqrt{-\det h_{ab}} \approx \sqrt{\left(1 - \frac{1}{c^2}\left(\partial_t \vec{Y}\right)^2\right)\left(1 + \left(\partial_x \vec{Y}\right)^2\right)}. \tag{19}$$

As in part (a), we may apply the Taylor expansion (10) to (19) to find

$$\sqrt{-\det h_{ab}} \approx 1 - \frac{1}{2c^2}\left(\partial_t \vec{Y}\right)^2 - \frac{1}{2}\left(\partial_x \vec{Y}\right)^2. \tag{20}$$

Finally, we will replace this expression (20) for $\sqrt{-\det h_{ab}}$ in the Nambu-Goto action. To make the comparison with the non-relativistic case clearer, I will also define the string tension

$$T_0 = \frac{1}{2\pi\alpha'}, \tag{21}$$

which allows us to write

$$S_{\text{NG}} = \int dt \int_0^\ell dx \left(-T_0 + \frac{1}{2}\frac{T_0}{c^2}\left(\partial_t \vec{Y}\right)^2 - \frac{T_0}{2}\left(\partial_x \vec{Y}\right)^2 \right). \tag{22}$$

You may recall from a course on waves and vibrations that the action for a classical, non-relativistic, massive string of tension T_0 and linear mass density μ_0 is

$$S_{\text{non-relativistic string}} = \int dt \int_0^\ell dx \left[\frac{1}{2}\mu_0 \left(\frac{\partial \vec{Y}}{\partial t}\right)^2 - \frac{1}{2}T_0 \left(\frac{\partial \vec{Y}}{\partial x}\right)^2 \right], \tag{23}$$

where x is the coordinate pointing tangentially along the string and $\vec{Y}$ represents the transverse coordinates.

Thus we see that the Nambu-Goto action reduces, in the non-relativistic limit, to the same action as that of an ordinary vibrating string (up to a total derivative term, $-T_0\ell$, which does not affect the equations of motion). Indeed, comparing (23) to (22), we see that the mass per unit length of the Nambu-Goto string is

$$\mu_{\text{NG}} = \frac{T_0}{c^2}, \tag{24}$$

which confirms our intuition that the total derivative term,

$$-T_0\ell = -\left(\frac{T_0}{c^2}\right)\ell c^2 = -\left(\mu_{\text{NG}}\ell\right)c^2 = -m_{\text{NG}}c^2, \tag{25}$$

is the "potential" due to the mass-energy of the string.

The Nambu-Goto string is sometimes said to be "massless", in the sense that it has no intrinsic rest mass – all of the non-kinetic mass-energy (25) comes from the potential energy due to stretching against the tension T_0.

Exercise 3. *Polchinski 1.3* 10 points

For world-sheets with boundary, show that

$$\chi = \frac{1}{4\pi}\int_M d\tau\, d\sigma\, (-\gamma)^{1/2}\, R + \frac{1}{2\pi}\int_{\partial M} ds\, k \tag{26}$$

is Weyl-invariant[1]. Here ds is the proper time along the boundary in the metric γ_{ab}, and k is the geodesic curvature of the boundary,

$$k = \pm t^a n_b \nabla_a t^b, \tag{27}$$

where t^a is a unit vector tangent to the boundary and n^a is an outward pointing unit vector orthogonal to t^a. The upper sign is for a timelike boundary and the lower sign for a spacelike boundary.

Solution 3.

Consider a Weyl transformation which sends

$$\gamma_{ab} \to \tilde{\gamma}_{ab} = e^{2\omega(\sigma,\tau)}\gamma_{ab} \tag{28}$$

for some function $\omega(\sigma, \tau)$ on the worldsheet.

We need to determine how each of the objects appearing in the definition (26) of χ transforms under a conformal change (28). I often refer to this Wikipedia section when considering geometric data under conformal maps. It tells us that, under a re-scaling (28) in $n = 2$ dimensions, the Ricci scalar transforms as

$$R \longrightarrow \tilde{R} = e^{-2\omega}\left(R - 2\nabla^\mu \partial_\mu \omega\right), \tag{29}$$

where ∇^μ is the covariant derivative with respect to the old metric.

Likewise, since we are in two dimensions, the measure transforms as

$$\begin{aligned}
\sqrt{-\det\left(\gamma_{ab}\right)} &\longrightarrow \sqrt{-\det\left(e^{-2\omega}\gamma_{ab}\right)} \\
&= \sqrt{-e^{-4\omega}\det\left(\gamma_{ab}\right)} \\
&= e^{-2\omega}\sqrt{-\det\left(\gamma_{ab}\right)},
\end{aligned} \tag{30}$$

since $\det(cM_{ab}) = c^n \det(M_{ab})$ for any $n \times n$ matrix M_{ab} (by the linearity of the determinant in each of the n rows or columns).

Alternatively, you could use Polchinski's equation (1.2.31), which tells you that

$$(-\tilde{\gamma})^{1/2}\,\tilde{R} = (-\gamma)^{1/2}\left(R - 2\nabla^2\omega\right), \tag{31}$$

rather than using the two separate results above.

With these formulas in hand, we can handle the transformation of the first term of (26), since

$$\frac{1}{4\pi}\int_M d\tau\, d\sigma\, (-\gamma)^{1/2}\left(R - 2\nabla^2\omega\right) = \frac{1}{4\pi}\left[\left(\int_M d\tau\, d\sigma\, (-\gamma)^{1/2}\, R\right) - 2\left(\int_M d\tau\, d\sigma\, \partial^a\left(\nabla_a\sqrt{-\gamma}\omega\right)\right)\right], \tag{32}$$

where we have used the result[2]

$$\nabla_m V^m = \frac{1}{\sqrt{-\gamma}}\partial_m\left(V^m\sqrt{-\gamma}\right). \tag{33}$$

In the closed string case, the second term in (32) vanished because it is the integral of a total divergence and we had no boundary. But in the open string case, the worldsheet does have a boundary ∂M; hence when we apply Stokes' theorem, it reduces to a surface term

$$-\frac{1}{2\pi}\int_{\partial M} ds\, n^a \partial_a \omega, \tag{34}$$

[1] Note that (26) is the Gauss-Bonnet theorem, which relates the Euler characteristic χ – a famous *topological* invariant – to an integral involving *geometrical* data about curvature. In short, topology = $\int$ geometry.

[2] Also on the Wikipedia page, under the section "Gradient, divergence, Laplace-Beltrami operator".

which we will need to cancel using the second term of (26).

Thus let's think about the transformation properties of the geodesic curvature k on the boundary. This k is defined in terms of *unit vectors* t^a and n^a on the boundary, but the notion of "unit vector" is metric-dependent: the condition

$$t^a t_a = t^a t^b \gamma_{ab} = \mp 1 \tag{35}$$

defines a unit vector (upper sign for a timelike boundary, lower sign for a spacelike boundary) with respect to the old metric γ_{ab}, but we need

$$\tilde{t}^a \tilde{t}^b \tilde{\gamma}_{ab} = \mp 1 \tag{36}$$

for a unit vector with respect to the new metric $\tilde{\gamma}_{ab} = e^{2\omega}\gamma_{ab}$. So the unit vectors must each be re-scaled as $t^a \to e^{-\omega} t^a$, $n^a \to e^{-\omega} n^a$. The versions of these vectors with downstairs indices, naturally, transform with the inverse factor of $e^{+\omega}$.

Thus the new geodesic curvature is

$$\begin{aligned} \tilde{k} &= \pm \tilde{t}^a \tilde{n}_b \tilde{\nabla}_a \tilde{t}^b \\ &= \pm t^a n_b \left(\partial_a \left(e^{-\omega} t^b \right) + \tilde{\Gamma}^b{}_{ac} \tilde{t}^c \right). \end{aligned} \tag{37}$$

We will need one more appeal to Wikipedia to see how the Christoffel symbols $\Gamma^b{}_{ac}$ transform under a conformal change:

$$\tilde{\Gamma}^b{}_{ac} = \Gamma^b{}_{ac} + \delta^b{}_a \partial_c \omega + \delta^b{}_c \partial_a \omega - \gamma_{ac} \partial^b \omega. \tag{38}$$

Thus

$$\begin{aligned} \tilde{\Gamma}^b{}_{ac} \tilde{t}^c &= \left(\Gamma^b{}_{ac} + \delta^b{}_a \left(\partial_c \omega \right) + \delta^b{}_c \partial_a \omega - \gamma_{ac} \partial^b \omega \right) \left(e^{-\omega} t^c \right) \\ &= \Gamma^b{}_{ac} \left(e^{-\omega} t^c \right) + \delta^b{}_a \left(\partial_c \omega \right) \left(e^{-\omega} t^c \right) + \left(e^{-\omega} t^b \right) \left(\partial_a \omega \right) - \gamma_{ac} \left(\partial^b \omega \right) \left(e^{-\omega} t^c \right). \end{aligned} \tag{39}$$

By assumption $n_b t^b = 0$ since the unit tangent and unit normal are orthogonal. So after contracting (39) against n_b, the third term vanishes. Then $\tilde{k}$ becomes

$$\begin{aligned} \tilde{k} &= \pm t^a n_b \left(\partial_a \left(e^{-\omega} t^b \right) + \tilde{\Gamma}^b{}_{ac} \tilde{t}^c \right) \\ &= \pm t^a n_b \left(\partial_a \left(e^{-\omega} t^b \right) + \Gamma^b{}_{ac} \left(e^{-\omega} t^c \right) + \delta^b{}_a \left(\partial_c \omega \right) \left(e^{-\omega} t^c \right) - \gamma_{ac} \left(\partial^b \omega \right) \left(e^{-\omega} t^c \right) \right) \\ &= e^{-\omega} k \pm t^a n_b \left(-e^{-\omega} \left(\partial_a \omega \right) t^b + \delta^b{}_a \left(\partial_c \omega \right) \left(e^{-\omega} t^c \right) - \gamma_{ac} \left(\partial^b \omega \right) \left(e^{-\omega} t^c \right) \right) \\ &= e^{-\omega} k \pm e^{-\omega} t^a n_b \left(-\gamma_{ac} \left(\partial^b \omega \right) t^c \right). \end{aligned} \tag{40}$$

In the last step, we have again used $t^a n_a = 0$. Now we may use $t^a t^c \gamma_{ac} = \mp 1$ to simplify the remaining term – tracking signs to find $(\mp)(\pm)(-1) = +1$ – and conclude that

$$\tilde{k} = e^{-\omega} \left(k + n^a \partial_a \omega \right). \tag{41}$$

Finally, we return to the boundary term of (26). We know how k transforms, and we know that $ds = \sqrt{\gamma_{ab} dx^a dx^b}$ changes as $ds \longrightarrow \tilde{ds} = \sqrt{e^{2\omega} \gamma_{ab} dx^a dx^b} = e^{\omega} ds$, so

$$\begin{aligned} \frac{1}{2\pi} \int_{\partial M} ds\, k \longrightarrow \frac{1}{2\pi} \int_{\partial M} \tilde{ds}\, \tilde{k} & \\ &= \frac{1}{2\pi} \int_{\partial M} e^{\omega} ds \, \left(e^{-\omega} \left(k + n^a \partial_a \omega \right) \right) \\ &= \left(\frac{1}{2\pi} \int_{\partial M} ds\, k \right) + \left(\frac{1}{2\pi} \int_{\partial M} ds\, n^a \partial_a \omega \right). \end{aligned} \tag{42}$$

Happily, the boundary piece picks up precisely the term needed to cancel (34) arising from the transformation of the bulk piece. We conclude that the quantity χ is also conformally invariant in the open string case, as desired.

Exercise 4. *Polchinski 1.5* 10 points

Extend the sum (1.3.32) to the 'twisted' case

$$\sum_{n=1}^{\infty}(n-\theta) \tag{43}$$

with θ a constant. That is, $k_\sigma = (n-\theta)\pi/\ell$. The answer is given in eq. (2.9.19). You should find that the cutoff-dependent term is independent of θ.

Solution 4.

Following Polchinski's treatment of the $\theta = 0$ case, we will regulate this divergent sum by introducing an exponential smoothing factor:

$$\sum_{n=1}^{\infty}(n-\theta) \longrightarrow \sum_{n=1}^{\infty}(n-\theta)\exp\left(-\epsilon\gamma_{\sigma\sigma}^{-1/2}\frac{(n-\theta)\pi}{\ell}\right). \tag{44}$$

First, let's remind ourselves of the strategy: because the exponential decays so quickly, the sum on the right side of (44) will converge for any finite $\epsilon > 0$. But on the other hand, as we take $\epsilon \to 0^+$, the terms converge pointwise to those of the original, un-regulated sum. Therefore, we might hope that summing the regulated expression (44) and taking $\epsilon \to 0^+$ in the result will give us data about the un-regulated sum. Unsurprisingly, though, taking $\epsilon \to 0^+$ gives a divergence. Thus we will retreat to "subtracting off the infinite piece" of the resulting divergent expression, hoping that the finite piece has physical meaning.[3]

Thus consider the general smoothed sum

$$\sum_{n=1}^{\infty}(n-\theta)\, e^{-c(n-\theta)\epsilon} \tag{45}$$

where c is some constant, which we will later set to $\gamma_{\sigma\sigma}^{-1/2}\frac{\pi}{\ell}$, and ϵ is a small parameter.

We use something similar to the "Feynman trick" of differentiating under an integral, but here we differentiate under a sum. That is, note that we may formally re-write each term in the sum (45) as

$$(n-\theta)\, e^{-c(n-\theta)\epsilon} = -\frac{1}{c}\frac{d}{d\epsilon}\left(e^{-c(n-\theta)\epsilon}\right). \tag{46}$$

If we believe that differentiation should commute with divergent series summation, we find

$$\begin{aligned}\sum_{n=1}^{\infty}(n-\theta)\, e^{-c(n-\theta)\epsilon} &= \sum_{n=1}^{\infty}\left(-\frac{1}{c}\right)\frac{d}{d\epsilon}\left(e^{-c(n-\theta)\epsilon}\right)\\ &= \left(-\frac{1}{c}\right)\frac{d}{d\epsilon}\sum_{n=1}^{\infty}\left(e^{-cn\epsilon}e^{c\theta\epsilon}\right)\\ &= \left(-\frac{1}{c}\right)\frac{d}{d\epsilon}\left[e^{c\theta\epsilon}\sum_{n=1}^{\infty}e^{-cn\epsilon}\right].\end{aligned} \tag{47}$$

Since $cn\epsilon > 0$ the expression $\sum_{n=1}^{\infty}e^{-cn\epsilon}$ is a geometric series whose sum is given by

$$\sum_{n=1}^{\infty}e^{-cn\epsilon} = \sum_{n=1}^{\infty}\left(e^{-c\epsilon}\right)^n = \frac{e^{-c\epsilon}}{1-e^{-c\epsilon}} = \frac{1}{e^{c\epsilon}-1}. \tag{48}$$

[3] Admittedly, this sentence sounds extremely unconvincing, especially to a mathematician. If you are skeptical about the meaningfulness of the finite piece, or wonder whether it depends on the choice of smoothing function, perhaps read this blog post by Terry Tao.

Using this in (47),

$$\sum_{n=1}^{\infty}(n-\theta)\,e^{-c(n-\theta)\epsilon} = \left(-\frac{1}{c}\right)\frac{d}{d\epsilon}\left[\frac{e^{c\theta\epsilon}}{e^{c\epsilon}-1}\right] \tag{49}$$

$$= \left(-\frac{1}{c}\right)\left(-\frac{ce^{c\epsilon+c\theta\epsilon}}{(e^{c\epsilon}-1)^2}+\frac{ce^{c\theta\epsilon}\theta}{e^{c\epsilon}-1}\right)$$

$$= \frac{e^{c\epsilon+c\theta\epsilon}}{(e^{c\epsilon}-1)^2}-\frac{\theta e^{c\theta\epsilon}}{e^{c\epsilon}-1}.$$

$$= \frac{e^{c\epsilon+c\theta\epsilon}-\theta e^{c\theta\epsilon}(e^{c\epsilon}-1)}{(e^{c\epsilon}-1)^2} \tag{50}$$

Now we take $\epsilon \to 0$ and extract the finite piece. There are a few ways to do this; I think it's instructive to see that they agree:

1. *First way.* The simplest strategy – which suffices for full credit on the problem set – is to ask Stephen Wolfram:

```
In[15]:= Series[(E^(c ε + c ε θ) - E^(c ε θ) θ (E^(c ε) - 1))/(E^(c ε) - 1)^2, {ε, 0, 0}]

Out[15]= 1/(c^2 ε^2) + 1/12 (-1 + 6 θ - 6 θ^2) + O[ε]^1
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2. *Second way.* If we want to see the result analytically, we'll need to get our hands dirty: one must find the divergent terms in the Laurent series of (50), subtract off the divergences to give a regular expression, and then take $\epsilon \to 0$ in the result to find the finite part.

   It's slightly easier to find the Laurent series for the quantity $\frac{e^{c\theta\epsilon}}{e^{c\epsilon}-1}$ in the first line (49), *before* differentiating. We see that, near $\epsilon = 0$, this expression blows up like $\frac{1}{\epsilon}$:

$$\frac{e^{c\theta\epsilon}}{e^{c\epsilon}-1} = \frac{1+c\theta\epsilon}{(1+c\epsilon)-1}+\mathcal{O}(1)$$

$$= \frac{1}{c\epsilon}+\mathcal{O}(1). \tag{51}$$

   Thus, the expression obtained by subtracting off the $\frac{1}{c\epsilon}$ divergence is purely regular and has a Taylor series around $\epsilon = 0$. We find the leading terms in this expansion as

$$\frac{e^{c\theta\epsilon}}{e^{c\epsilon}-1}-\frac{1}{c\epsilon} = \frac{c\epsilon e^{c\theta\epsilon}-(e^{c\epsilon}-1)}{c\epsilon\,(e^{c\epsilon}-1)}$$

$$= \left[\frac{c\epsilon e^{c\theta\epsilon}-(e^{c\epsilon}-1)}{c\epsilon\,(e^{c\epsilon}-1)}\right]_{\epsilon=0} + \left[\frac{d}{d\epsilon}\left(\frac{c\epsilon e^{c\theta\epsilon}-(e^{c\epsilon}-1)}{c\epsilon\,(e^{c\epsilon}-1)}\right)\right]_{\epsilon=0}\epsilon+\mathcal{O}(\epsilon^2). \tag{52}$$

   The first term, in the limit near $\epsilon = 0$, is given by

$$\lim_{\epsilon\to 0}\left[\frac{c\epsilon e^{c\theta\epsilon}-(e^{c\epsilon}-1)}{c\epsilon\,(e^{c\epsilon}-1)}\right] = \lim_{\epsilon\to 0}\left[\frac{c\epsilon\left(1+c\theta\epsilon+\frac{1}{2}c^2\theta^2\epsilon^2\right)-\left(c\epsilon+\frac{1}{2}c^2\epsilon^2\right)}{c^2\epsilon^2}\right]$$

$$= \lim_{\epsilon\to 0}\left[\frac{c^2\epsilon^2\left(\theta-\frac{1}{2}\right)}{c^2\epsilon^2}\right]$$

$$= \theta-\frac{1}{2}. \tag{53}$$

The second piece, again near $\epsilon = 0$, is

$$\left[\frac{d}{d\epsilon}\left(\frac{c\epsilon e^{c\theta\epsilon} - (e^{c\epsilon}-1)}{c\epsilon\,(e^{c\epsilon}-1)}\right)\right]_{\epsilon=0} = \lim_{\epsilon\to 0}\left[\frac{c^2\epsilon^2 e^{c\theta\epsilon}\left(\theta e^{c\epsilon} - e^{c\epsilon} - \theta\right) + \left(e^{c\epsilon}-1\right)^2}{c\epsilon^2\left(e^{c\epsilon}-1\right)^2}\right]$$
$$= \lim_{\epsilon\to 0}\left\{\frac{1}{c^2\epsilon^2(c\epsilon)^2}\left[c^2\epsilon^2\left(1 + c\theta\epsilon + \frac{1}{2}c^2\theta^2\epsilon^2\right)\left((\theta-1)\left(1 + c\epsilon + \frac{1}{2}c^2\epsilon^2\right) - \theta\right) + \left(c\epsilon + \frac{1}{2}c^2\epsilon^2 + \frac{1}{6}c^3\epsilon^3\right)^2\right]\right\} \tag{54}$$

In the numerator, the terms proportional to $c^2\epsilon^2$ cancel between the first and second terms, and likewise with the $c^3\epsilon^3$ pieces. At order ϵ^4 in the numerator, we get $\frac{1}{2}c^4\epsilon^4\left(\theta^2 - \theta - 1\right)$ in the first term and $\frac{7}{12}c^4\epsilon^4$ in the second term.

(Note that the factor of $\frac{7}{12}$ came from expanding $\left(c\epsilon + \frac{1}{2}(c\epsilon)^2 + \frac{1}{6}(c\epsilon)^3\right)^2$, which has two contributions to the ϵ^4 term: one when we get two copies of the middle term, and one when we get one copy of the first term and one copy of the third.)

Canceling with the $c^3\epsilon^4$ in the bottom, we find

$$\left[\frac{d}{d\epsilon}\left(\frac{c\epsilon e^{c\theta\epsilon} - (e^{c\epsilon}-1)}{c\epsilon\,(e^{c\epsilon}-1)}\right)\right]_{\epsilon=0} = c\left(\frac{1}{12} - \frac{\theta}{2} + \frac{\theta^2}{2}\right). \tag{55}$$

Putting this together, (49) is

$$\sum_{n=1}^{\infty}(n-\theta)\,e^{-c(n-\theta)\epsilon} = \left(-\frac{1}{c}\right)\frac{d}{d\epsilon}\left[\frac{e^{c\theta\epsilon}}{e^{c\epsilon}-1}\right]$$
$$= \left(-\frac{1}{c}\right)\frac{d}{d\epsilon}\left[\frac{1}{c\epsilon} + \left(\theta - \frac{1}{2}\right) + c\epsilon\left(\frac{1}{12} - \frac{\theta}{2} + \frac{\theta^2}{2}\right) + \mathcal{O}(\epsilon^2)\right]$$
$$= \frac{1}{c^2\epsilon^2} + \left(-\frac{1}{12} - \frac{\theta}{2} + \frac{\theta^2}{2}\right) + \mathcal{O}(\epsilon). \tag{56}$$

Note that one could instead analyze the expression (50), after differentiating, which gives a calculation similar to the second way above (one expands the numerator and denominator of (50) to fourth order in ϵ, perhaps after subtracting off the $\frac{1}{\epsilon^2}$ pole if you'd like, and then finds the same result).

Regardless of which approach you prefer, we conclude

$$\sum_{n=1}^{\infty}(n-\theta)e^{-c(n-\theta)\epsilon} = \frac{1}{c^2\epsilon^2} + \frac{1}{12}\left(-1 + 6\theta - 6\theta^2\right) + \mathcal{O}(\epsilon), \tag{57}$$

or, extracting the regular part when $\epsilon \to 0$,

$$\operatorname{reg}\left[\sum_{n=1}^{\infty}(n-\theta)\right] = \frac{1}{12}\left(-1 + 6\theta - 6\theta^2\right). \tag{58}$$

The cutoff-dependent first term, after replacing $c = \gamma_{\sigma\sigma}^{-1/2}\frac{\pi}{\ell}$, is independent of θ, as expected. As in the un-twisted sum, this term is proportional to the length of the string and can be canceled by adding a cosmological-constant-style counterterm of the form $\int d^2\sigma\,\sqrt{-\gamma}$.

Our result (58) also agrees with Polchinski's equation (2.9.19), after some algebra:

$$\frac{1}{24} - \frac{1}{8}(2\theta-1)^2 = \frac{1}{24} - \frac{1}{8}\left(4\theta^2 - 4\theta + 1\right) =$$
$$= \frac{1}{24} - \frac{3}{24} + \frac{\theta}{2} - \frac{\theta^2}{2}$$
$$= \frac{1}{12}\left(-1 + 6\theta - 6\theta^2\right). \tag{59}$$

In the following exercises one has the usual open or closed string boundary conditions (Neumann or periodic) on X^μ for $\mu = 0, \cdots, 24$ but a different boundary condition on X^{25}. Each of these has an important physical interpretation, and will be developed in detail in chapter 8. Find the mode expansion, the mass spectrum, and (for the closed string) the constraint from σ-translation invariance in terms of the occupation numbers. In some cases you need the result of exercise 1.5.

Note: In the solutions to Polchinski 1.8 and 1.9, I will set $\ell = 2\pi$ to avoid cluttering the formulas.

Exercise 5. *Polchinski 1.8* 10 points

Closed strings with

$$X^{25}(\tau, \sigma + \ell) = X^{25}(\tau, \sigma) + 2\pi R \tag{60}$$

with R a constant. This is a winding string in a toroidal (periodic) compactification. In this case p^{25} must be a multiple of $1/R$.

Solution 5.

We recall from lecture that, after using reparameterization invariance to put the worldsheet metric into conformal gauge $\gamma_{ab} = e^{2\phi}\eta_{ab}$ and then using Weyl invariance to set $\phi = 0$, the equations of motion for the string embedding coordinates $X^\mu(\sigma, \tau)$ reduce to the simple wave equations

$$\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right) X^\mu = 0. \tag{61}$$

(Of course, this must be supplemented by the constraints arising from the vanishing of the stress tensor, namely $\dot{X}^\mu X'_\mu = 0 = \left(\dot{X}^\mu + X^{\mu\prime}\right)^2$, although this will not be relevant here.)

We know that solutions to the wave equation (61) have the general form

$$X^\mu(\sigma, \tau) = X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma). \tag{62}$$

For the coordinates $X^0, \cdots, X^{24}$, we have the usual periodic boundary condition $X^\mu(\sigma + 2\pi, \tau) = X^\mu(\sigma, \tau)$, which is solved by the familiar mode expansion

$$X^i(\tau, \sigma) = x^i + \alpha' p^i \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{e^{-in\tau}}{n} \left(\alpha_n^i e^{in\sigma} + \tilde{\alpha}_n^i e^{-in\sigma}\right), \tag{63}$$

where $i = 0, \cdots, 24$.

Now we consider X^{25}, which satisfies the new boundary condition

$$X^{25}(\tau, \sigma + 2\pi) = X^{25}(\sigma, \tau) + 2\pi R. \tag{64}$$

I claim that this boundary condition still admits a similar expansion, albeit with a new term linear in σ. Let's show this explicitly.

Let $\sigma^\pm = \tau \pm \sigma$. Then the boundary condition (64) can be written as

$$X_L^{25}(\sigma^+ + 2\pi) - X_L^{25}(\sigma^+) = X_R^{25}(\sigma^-) - X_R^{25}(\sigma^- - 2\pi) + 2\pi R. \tag{65}$$

Differentiating both sides with respect to σ^+, treated as an independent variable from σ^-, gives $X_L^{25\prime}(\sigma^+ + 2\pi) = X_L^{25\prime}(\sigma^+)$, which means that the derivative $X_L^{25\prime}$ is a periodic function of its argument. Similarly, differentiating both sides with respect to σ^- shows that $X_R^{25\prime}$ is a periodic function of its argument.

If $X_L^{25\prime}$ and $X_R^{25\prime}$ are periodic functions of σ^+ and σ^-, respectively, then they can be written as Fourier expansions:

$$\begin{aligned} X_L^{25\prime}(\sigma^+) &= \sum_{n=-\infty}^{\infty} \tilde{\alpha}_n^{25} e^{-in\sigma^+}, \\ X_R^{25\prime}(\sigma^-) &= \sum_{n=-\infty}^{\infty} \alpha_n^{25} e^{-in\sigma^-}. \end{aligned} \tag{66}$$

We integrate the two equations in (66) to obtain

$$\begin{aligned} X_L^{25}(\sigma^+) &= \frac{1}{2}x_0^{25} + \frac{1}{2}\alpha' p_L^{25}\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\tilde{\alpha}_n^{25}}{n}e^{-in\sigma^+}, \\ X_R^{25}(\sigma^-) &= \frac{1}{2}x_0^{25} + \frac{1}{2}\alpha' p_R^{25}\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^{25}}{n}e^{-in\sigma^-}. \end{aligned} \tag{67}$$

I have allowed the momenta p_L^{25} and p_R^{25} to be different; as we will see, the boundary condition actually requires this to be the case. If we plug our left-moving and right-moving mode expansions (67) into the boundary condition, written in the form (65), we find

$$\begin{aligned} \frac{1}{2}\alpha' p_L^{25}\left(\sigma^+ + 2\pi\right) - \frac{1}{2}\alpha' p_L^{25}\sigma^+ &= \frac{1}{2}\alpha' p_R^{25}\sigma^- - \frac{1}{2}\alpha' p_R^{25}\left(\sigma^- - 2\pi\right) + 2\pi R. \\ \implies p_L^{25} &= p_R^{25} + \frac{2}{\alpha'}R. \end{aligned} \tag{68}$$

Aha! The left-moving and right-moving momenta differ precisely because of the winding around the compact coordinate X^{25}. Now we add X_L^{25} and X_R^{25} to obtain the full mode expansion for X^{25}, namely

$$\begin{aligned} X^{25}(\tau,\sigma) &= x_0^{25} + \frac{1}{2}\alpha'\left(p_R^{25}\left(\tau-\sigma\right) + \left(p_R^{25} + \frac{2}{\alpha'}R\right)(\tau+\sigma)\right) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}\left(\tilde{\alpha}_n^{25}e^{-in\sigma} + \alpha_n^{25}e^{in\sigma}\right) \\ &\equiv x_0^{25} + \alpha' p^{25}\tau + R\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}\left(\tilde{\alpha}_n^{25}e^{-in\sigma} + \alpha_n^{25}e^{in\sigma}\right). \end{aligned} \tag{69}$$

In the last step, I have chosen to call p_R^{25} simply p^{25}, since this is the coefficient appearing in the term of X^{25} which is linear in τ.

Therefore, we have found that the oscillator expansion for X^{25} has picked up a term linear in σ, which guarantees that $X^{25}(\tau,\sigma+2\pi) = X^{25}(\tau,\sigma) + 2\pi R$.

Quantizing the theory with this new boundary conditions proceeds almost identically to the case with usual periodic boundary conditions; all that has changed in our mode expansion (69) is the addition of a commuting number, $R\sigma$, which does not affect the commutation relations of the modes nor the normal-ordering constant. Thus we promote the Fourier coefficients α_n, $\tilde{\alpha}_m$ to operators which satisfy

$$\begin{aligned} \left[\alpha_m^i, \alpha_n^j\right] &= m\delta^{ij}\delta_{m,-n} = \left[\tilde{\alpha}_m^i, \tilde{\alpha}_n^j\right], \\ \left[\alpha_m^i, \tilde{\alpha}_n^j\right] &= 0. \end{aligned} \tag{70}$$

To find the mass formula, we use the spacetime identity $M^2 = p^\mu p_\mu = 2p^+p^- - p^ip^i$, where $p^- = H$ is the Hamiltonian. The momenta $p^i = \frac{p^+}{\ell}\int_0^\ell d\sigma\,\left(\partial_\tau X^i(\sigma,\tau)\right)$ are unchanged by our new boundary condition, since adding a term linear in σ will not change the derivative of X^{25} with respect to τ. Thus we need only find the change in the Hamiltonian,

$$H = \frac{1}{2\alpha' p^+}\int_0^{2\pi} d\sigma\,\left(2\pi\alpha'\Pi^i\Pi^i + \frac{1}{2\pi\alpha'}\partial_\sigma X^i\partial_\sigma X^i\right). \tag{71}$$

We see that the $\partial_\sigma X^i\partial_\sigma X^i$ term will be changed due to the new term $R\sigma$ appearing in our expansion (69) for X^{25}. With this added term, we find

$$H = \frac{p^ip^i}{2p^+} + \frac{1}{p^+\alpha'}\left(\sum_{n=1}^{\infty}\left(\alpha_{-n}^i\alpha_n^i + \tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i\right) - 2\right) + \frac{R^2}{2p^+\alpha'^2}, \tag{72}$$

where we have used the result that the normal-ordering constant is $A = -1$ in the critical dimension $D = 26$. Recognizing the appearance of the number operators N and $\tilde{N}$, we find

$$\left(M^2\right)_{26d} = \frac{2}{\alpha'}\left(N + \tilde{N} - 2 + \frac{R^2}{2\alpha'}\right). \tag{73}$$

On the other hand, for an observer living in the 25-dimensional space $X^0, \cdots, X^{24}$ but who cannot see the circular dimension X^{25} (perhaps because R is very small), the mass formula $M^2 = 2p^+p^- - \sum_{i=2}^{24} p^i p^i$ excludes p^{25} in the sum of the second term. Such an observer measures

$$\begin{aligned}
\left(M^2\right)_{25d} &= \frac{2}{\alpha'}\left(N + \tilde{N} - 2 + \frac{R^2}{2\alpha'} + \frac{\alpha'}{2}\left(p^{25}\right)^2\right) \\
&= \frac{2}{\alpha'}\left(N + \tilde{N} - 2 + \frac{R^2}{2\alpha'} + \frac{\alpha' n^2}{2R^2}\right),
\end{aligned} \tag{74}$$

which makes it clearer that the winding momentum around the circle of radius R and physical momentum $p^{25} = \frac{n}{R}$ have similar contributions to the mass. In fact, transforming the radius as $R \to \tilde{R} = \frac{\alpha'}{R}$ when $n = 1$ leaves the mass unchanged (this observation leads to *T-duality*).

Finally, we would like to find the level-matching condition which arises from invariance under σ-translations. The conserved momentum associated with shifts in σ is

$$\begin{aligned}
P &= -\int_0^{2\pi} d\sigma\, \Pi^i \partial_\sigma X^i \\
&= -\left[\sum_{n=1}^{\infty}\left(\alpha^i_{-n}\alpha^i_n - \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n\right) + p^{25}R\right] \\
&= -\left(N - \tilde{N} + p^{25}R\right),
\end{aligned} \tag{75}$$

where the extra piece came from differentiating the new $R\sigma$ term in X^{25}. Thus the new level-matching condition is

$$N - \tilde{N} = p^{25}R \overset{!}{=} n, \tag{76}$$

where in the last step we have used that $p^{25} = \frac{n}{R}$ must be an integer. More generally, if the boundary condition had been $X^{25}(\tau, \sigma + 2\pi) = X^{25}(\tau, \sigma) + 2\pi Rm$ for some integer m, we would have found that $N - \tilde{N} = nm$.

In the following exercises one has the usual open or closed string boundary conditions (Neumann or periodic) on X^μ for $\mu = 0, \cdots, 24$ but a different boundary condition on X^{25}. Each of these has an important physical interpretation, and will be developed in detail in chapter 8. Find the mode expansion, the mass spectrum, and (for the closed string) the constraint from σ-translation invariance in terms of the occupation numbers. In some cases you need the result of exercise 1.5.

Exercise 6. *Polchinski 1.9* 10 points

Closed strings with

$$X^{25}(\tau, \sigma + \ell) = -X^{25}(\sigma, \tau). \tag{77}$$

This is a twisted string in orbifold compactification.

Solution 6.

First, we seek the oscillator expansion consistent with the orbifold identification (77). Let's go slowly: again, we note that solutions to $\partial_\alpha \partial^\alpha X^{25} = 0$ can be written as

$$X^{25}(\tau, \sigma) = X_L^{25}(\sigma^+) + X_R^{25}(\sigma^-). \tag{78}$$

Our boundary condition (77) requires

$$X_L^{25}(\sigma^+ + 2\pi) + X_R^{25}(\sigma^- - 2\pi) = -\left(X_L^{25}(\sigma^+) + X_R^{25}(\sigma^-)\right). \tag{79}$$

As in problem (1.8), we take derivatives with respect to σ^+ and σ^-. This yields the two equations

$$X_L^{25\prime}(\sigma^+ + 2\pi) = -X_L^{25\prime}(\sigma^+) \quad , \quad X_R^{25\prime}(\sigma^- + 2\pi) = -X_R^{25\prime}(\sigma^-). \tag{80}$$

We see that the derivatives $X_L^{25\prime}$ and $X_R^{25\prime}$ reverse sign when their arguments advance by 2π. The most general function with this property can be written as a sum of exponentials $\exp(ik\sigma^\pm)$ where k is a half-integer, that is,

$$\begin{aligned} X_L^{25\prime}(\sigma^+) &= \sqrt{\frac{\alpha'}{2}} \sum_{n \text{ odd}} \tilde{\alpha}^{25}_{\frac{n}{2}} \exp\left(-i\frac{n}{2}\sigma^+\right), \\ X_R^{25\prime}(\sigma^-) &= \sqrt{\frac{\alpha'}{2}} \sum_{n \text{ odd}} \alpha^{25}_{\frac{n}{2}} \exp\left(-i\frac{n}{2}\sigma^-\right). \end{aligned} \tag{81}$$

Integrating (81), we find

$$\begin{aligned} X_L^{25}(\sigma^+) &= i\sqrt{\frac{\alpha'}{2}} \sum_{n \text{ odd}} \frac{2}{n} \tilde{\alpha}^{25}_{\frac{n}{2}} e^{-i\frac{n}{2}\sigma^+}, \\ X_R^{25}(\sigma^-) &= i\sqrt{\frac{\alpha'}{2}} \sum_{n \text{ odd}} \frac{2}{n} \alpha^{25}_{\frac{n}{2}} e^{-i\frac{n}{2}\sigma^-}. \end{aligned} \tag{82}$$

Note that, if we had included any constant or linear terms in X_L and X_R, the boundary condition (77) would have forced the two contributions to cancel when we sum to get $X^{25}(\tau, \sigma)$: otherwise we cannot have $X^{25}(\tau, \sigma + 2\pi) = -X^{25}(\tau, \sigma)$. We therefore set these terms equal to zero, without loss of generality, since we care only about the sum X^{25} rather than about X_L^{25} and X_R^{25} individually.

Thus, combining the left-moving and right-moving results to form $X^{25}(\tau, \sigma) = X_L(\sigma^+) + X_R(\sigma^-)$, we have

$$X^{25}(\tau, \sigma) = i\sqrt{\frac{\alpha'}{2}} \sum_{n \text{ odd}} \frac{2}{n} e^{-i\frac{n}{2}\tau} \left(\tilde{\alpha}^{25}_{\frac{n}{2}} e^{-i\frac{n}{2}\sigma} + \alpha^{25}_{\frac{n}{2}} e^{i\frac{n}{2}\sigma}\right). \tag{83}$$

Next, we will quantize the theory. The new half-integral modes have the expected commutation relations,

$$\begin{aligned} \left[\alpha^{25}_{\frac{n}{2}}, \alpha^{25}_{\frac{m}{2}}\right] &= \frac{m}{2}\delta_{m,-n} = \left[\tilde{\alpha}^{25}_{\frac{n}{2}}, \tilde{\alpha}^{25}_{\frac{m}{2}}\right], \\ \left[\alpha^{25}_{\frac{n}{2}}, \tilde{\alpha}^{25}_{\frac{m}{2}}\right] &= 0. \end{aligned} \tag{84}$$

However, when we compute the normal-ordering constant associated with the ambiguity of operator ordering in the expression

$$\sum_{i\neq 25}\left(\sum_{n=1}^{\infty}\alpha^i_{-n}\alpha^i_n\right)+\sum_{k\text{ odd}}\alpha^{25}_{-\frac{k}{2}}\alpha^{25}_{\frac{k}{2}}, \tag{85}$$

the result will be modified because of the second term, which is now a sum over half-integers.

Thus we will need to compute the sum of all positive half-integers (or, equivalently, half of the sum of all odd positive integers). For your amusement, I present two ways to do this: the first is completely non-rigorous but turns out to be correct, and the second uses the results of exercise 1.5

1. The non-rigorous way is to split the sum over all integers into even and odd terms as

$$\sum_{k=1}^{\infty}k=\sum_{k\geq 1,\text{ odd}}k+\sum_{k\geq 2\text{ even}}k=\sum_{k\text{ odd}}k+2\sum_{k=1}^{\infty}k. \tag{86}$$

But we "know" that $\sum_{k=1}^{\infty}k=-\frac{1}{12}$, so equation (86) becomes

$$-\frac{1}{12}=\sum_{k\text{ odd}}k-\frac{1}{6}, \tag{87}$$

from which we conclude

$$\sum_{k\text{ odd}}k=+\frac{1}{12}, \tag{88}$$

or, dividing by 2,

$$\sum_{k\text{ half-integer}}k=\frac{1}{24}. \tag{89}$$

You may reasonably question whether this works, since re-arranging the terms of a series is not guaranteed to preserve its convergence properties (as it turns out, this is allowed for the regulated sums we consider here).

2. The more rigorous way is to appeal to exercise 1.5, where we showed (in equation (58)) that

$$\sum_{n=1}^{\infty}(n-\theta)=\frac{1}{12}\left(-1+6\theta-6\theta^2+\cdots\right).$$

Setting $\theta=\frac{1}{2}$, this gives us the desired sum over half-integers:

$$\sum_{n=1}^{\infty}\left(n-\frac{1}{2}\right)=\frac{1}{2}+\frac{3}{2}+\frac{5}{2}+\cdots=\frac{1}{12}\left(-1+\frac{6}{2}-\frac{6}{4}\right)=\frac{1}{24}. \tag{90}$$

With the exception of the new normal-ordering constant, the usual analysis of the mass-squared goes through. One finds

$$M^2=\frac{\alpha'}{2}\left(N+\tilde{N}-A-\tilde{A}\right). \tag{91}$$

The normal-ordering constants A and $\tilde{A}$ have contributions from twenty-three directions in the un-twisted directions, each of which adds the usual $\frac{1}{2}\cdot\left(-\frac{1}{12}\right)$, and one contribution from the new twisted direction, which adds $\frac{1}{4}\cdot\left(\frac{1}{12}\right)$.

More explicitly, recall that the normal-ordering constant for the integrally-moded fields came from the ambiguity in the order of operators appearing in the Hamiltonian, namely

$$A=\frac{1}{2}\sum_{i\neq 25,p\in\mathbb{Z}}\alpha^i_p\alpha^i_{-p}+\frac{1}{2}\sum_{k\text{ odd}}\alpha^{25}_{\frac{k}{2}}\alpha^{25}_{-\frac{k}{2}}. \tag{92}$$

The commutator of the integrally-moded α's is $[\alpha^i_n, \alpha^j_m] = n\delta_{m,-n}\delta^{ij}$, while the commutator of the half-integrally-moded α's is $[\alpha^i_{\frac{k}{2}}, \alpha^j_{-\frac{l}{2}}] = \frac{k}{2}\delta_{k,-l}$. So

$$A = \frac{1}{2}\sum_{i\neq 25, p\in\mathbb{Z}} n + \frac{1}{2}\sum_{k\text{ odd}} \frac{k}{2}. \tag{93}$$

There are 23 transverse values of i in the first sum, excluding 25, so we get $23\cdot\frac{1}{2}\cdot\left(-\frac{1}{12}\right)$. In the second sum, we get $\frac{1}{2}$ times the sum of half-integers, which we've shown to be $\frac{1}{24}$. Thus

$$A = \tilde{A} = 23\cdot\left(-\frac{1}{24}\right) + 1\cdot\left(\frac{1}{48}\right) = \frac{15}{16}, \tag{94}$$

and likewise for $\tilde{A}$. So altogether,

$$M^2 = \frac{\alpha'}{2}\left(N + \tilde{N} - \frac{15}{8}\right). \tag{95}$$

To conclude, we will find level-matching constraint on N and $\tilde{N}$ arising from σ-translation invariance. We compute

$$\begin{aligned} P &= -\int_0^{2\pi} d\sigma\, \Pi^i \partial_\sigma X^i \\ &= -\left[\sum_{n=1}^{\infty}\sum_{i\neq 25}\left(\alpha^i_{-n}\alpha^i_n - \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n\right) + \sum_{k\text{ odd}}\left(\alpha^{25}_{-\frac{n}{2}}\alpha^{25}_{\frac{n}{2}} - \tilde{\alpha}^{25}_{-\frac{n}{2}}\tilde{\alpha}^{25}_{\frac{n}{2}}\right)\right]. \end{aligned} \tag{96}$$

Thus the level-matching condition is still

$$N = \tilde{N}, \tag{97}$$

although now the occupation numbers N and $\tilde{N}$ include contributions from the half-integrally-moded oscillators in the 25 direction.