Grading.

The maximum score on this problem set was 10 $\frac{\text{points}}{\text{problem}}$ $\cdot$ 8 problems = 80 points. (Here I count "problem 1", which includes Polchinski 2.11 - 2.13, as three separate problems.)

Email cferko@uchicago.edu with questions or corrections.

Exercise 1. *Polchinski 2.11* 10 points

Evaluate the central charge in the Virasoro algebra for X^μ by calculating

$$L_m \left(L_{-m} |0;0\rangle \right) - L_{-m} \left(L_m |0;0\rangle \right) \tag{1}$$

Solution 1.

We wish to explicitly compute the action of the commutator $[L_m, L_{-m}]$ on the state $|0;0\rangle$, using the expansion of L_m in modes α_n (or the contour integral expression which defines L_m in terms of the stress tensor, but I will follow the former strategy here). This calculation should tell us the central charge of the free boson theory because the Virasoro algebra tells us that

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} \left(m^3 - m \right) \delta_{m,-n}, \tag{2}$$

and hence the commutator $[L_m, L_{-m}]$ should give us an operator which is the identity multiplied by $\frac{c}{12} \left(m^3 - m \right)$.

We have seen (for instance, in equation (2.7.6) of Polchinski) that the Virasoro operator L_m has the mode expansion

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha^\mu_{m-n} \alpha_{n,\mu}, \tag{3}$$

so long as $m \neq 0$, in which case there is an ordering ambiguity. However, in the case $m = 0$, the calculation of interest (1)is trivial, since $L_0^2 - L_0^2 = 0$ on any state. Thus we will restrict attention to Virasoro generators L_m with $m \neq 0$.

In fact, since $[L_{-m}, L_m] = -[L_m, L_{-m}]$, it is enough to compute the commutator when m is positive; to find the result for $-m$, we simply multiply the result by -1.

Thus for positive m, we compute

$$\begin{aligned} [L_m, L_{-m}] |0;0\rangle &= L_m \left(L_{-m} |0;0\rangle \right) - L_{-m} \left(L_m |0;0\rangle \right) \\ &= \frac{1}{4} \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left(\alpha^\mu_{m-n} \alpha_{n,\mu} \alpha^\nu_{-m-k} \alpha_{k,\nu} - \alpha^\mu_{-m-k} \alpha_{k,\mu} \alpha^\nu_{m-n} \alpha_{n,\nu} \right) |0;0\rangle. \end{aligned} \tag{4}$$

Now, since $\alpha_{i,\mu}$ annihilates the vacuum for all $i \geq 0$, the second term in (4) appears to be non-vanishing only for $n < 0$, since for such n we have $\alpha_{n,\mu}$ acting on $|0;0\rangle$. However, we then have α^μ_{m-n} acting on the result, which will always be a positive oscillator with an index different from n, which annihilates the state. Thus the second term in (4) vanishes identically, and we need only compute the first term.

Here we apply similar reasoning: the first term contains a pair of oscillators $\alpha^\nu_{-m-k} \alpha_{k,\nu}$ acting on $|0;0\rangle$, so that $\alpha_{k,\nu}$ annihilates the state unless $k < 0$ and α^ν_{-m-k} annihilates the result unless $-m - k < 0$ or $k > -m$, so the only surviving terms are

$$[L_m, L_{-m}] |0;0\rangle = \frac{1}{4} \left[\sum_{n=-\infty}^{\infty} \sum_{k=-m+1}^{-1} \left(\alpha^\mu_{m-n} \alpha_{n,\mu} \alpha^\nu_{-m-k} \alpha_{k,\nu} \right) \right] |0;0\rangle. \tag{5}$$

We now use the commutation relations

$$[\alpha^\mu_m, \alpha^\nu_n] = m \eta^{\mu\nu} \delta_{m,-n} \tag{6}$$

from our quantization of the bosonic string back in equation (1.4.16). We see that the product $\alpha^\mu_{m-n} \alpha_{n,\mu}$ will always annihilate $|0;0\rangle$, since the second operator annihilates the vacuum unless

$n < 0$, in which case $m - n > 0$ so the first operator annihilates the result. Thus we would like to commute both of these operators past the others. We have

$$\alpha_{n,\mu}\alpha^{\nu}_{-m-k} = \alpha^{\nu}_{-m-k}\alpha_{n,\mu} + n\delta_{\mu}{}^{\nu}\delta_{n,-m-k}, \tag{7}$$

which allows us to push $\alpha_{n,\mu}$ past α^{ν}_{-m-k}, and then

$$\alpha_{n,\mu}\alpha_{k,\nu} = \alpha_{k,\nu}\alpha_{n,\mu} + n\eta_{\mu\nu}\delta_{n,-k} \tag{8}$$

allows us to push it all the way to the right. Similarly, to push α^{μ}_{m-n} to the right, we first use

$$\alpha^{\mu}_{m-n}\alpha^{\nu}_{-m-k} = \alpha^{\nu}_{-m-k}\alpha^{\mu}_{-m-n} + (m-n)\,\delta_{n,-k}\eta^{\mu\nu} \tag{9}$$

and

$$\alpha^{\mu}_{m-n}\alpha_{k,\nu} = \alpha_{k,\nu}\alpha^{\mu}_{m-n} + (m-n)\,\delta_{m-n,-k}\delta^{\mu}{}_{\nu}. \tag{10}$$

After applying the four commutation relations (7) - (10), we are left with one oscillator term which is guaranteed to yield zero by the argument above, and four constant terms from the commutators which can be grouped as follows:

$$[L_m, L_{-m}]\,|0;0\rangle = \frac{1}{4}\sum_{n=-\infty}^{\infty}\sum_{k=-m+1}^{-1}\left(n(m-n)\delta_{n,-k}\eta^{\mu\nu}\eta_{\mu\nu} + n(m-n)\delta_{m-n,-k}\delta^{\mu}{}_{\nu}\delta_{\mu}{}^{\nu}\right)|0;0\rangle. \tag{11}$$

Using the delta functions to collapse the sums over n, and that $\eta^{\mu\nu}\eta_{\mu\nu} = D = \delta^{\mu}{}_{\nu}\delta_{\mu}{}^{\nu}$, this is

$$\begin{aligned}[L_m, L_{-m}]\,|0;0\rangle &= \frac{1}{4}\sum_{k=-m+1}^{-1}\left((-k)(m+k)D + (m+k)(-k)D\right)|0;0\rangle \\ &= -\frac{D}{2}\sum_{k=-m+1}^{-1}(k)(m+k)|0;0\rangle \end{aligned} \tag{12}$$

This is a simple sum which we can evaluate with formulas for sums of integers and squares, but I am quite lazy so I will ask Stephen Wolfram instead:

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In[16]:= FullSimplify[∑_{k=-m+1}^{-1} (-k) (m + k)]

Out[16]= 1/6 m (-1 + m^2)
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Thus we are left with

$$[L_m, L_{-m}]\,|0;0\rangle = \frac{D}{12}\left(m^3 - m\right)|0;0\rangle, \tag{13}$$

which is valid for any $m > 1$. But, as we have argued above, the result for negative m can be obtained by antisymmetry of the commutator, and both sides of (13) vanish for $m = 0$, so our conclusion is actually valid for all m.

Finally, comparison with the Virasoro algebra (2) identifies the central charge as

$$c = D, \tag{14}$$

as we expect for a theory of D free bosons X^{μ}.

Exercise 2. *Polchinski 2.12* 10 points
Use the OPE and the contour results (2.6.14) and (2.6.15) to derive the commutators (2.7.5) and (2.7.17).

Solution 2.
In this problem, we wish to convert from an OPE to the corresponding commutation relations using Polchinski's equation (2.6.14), which tells us that given two charges $Q_i = \oint_{\mathcal{C}} \frac{dz}{2\pi i} j_i(z)$ obtained from currents j_i, one has the commutator

$$[Q_1, Q_2] = \oint_{\mathcal{C}} \frac{dw}{2\pi i} \operatorname{Res}_{z\to w} j_1(z) j_2(w). \tag{15}$$

First we would like to apply (15) to find the commutation relations for the α^μ_m, which are the modes of $\partial X^\mu(z)$ in the sense that

$$\alpha^\mu_m = \sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi} z^m \partial X^\mu(z). \tag{16}$$

Note that the definitions of the currents Q_i includes a factor of i in the denominator of the contour integral, while the definition of the modes in (16) does not, so we must define the corresponding currents as

$$j^\mu_m = \sqrt{\frac{2}{\alpha'}} i z^m \partial X^\mu(z) \tag{17}$$

to make the conventions agree.

This gives

$$[\alpha^\mu_m, \alpha^\nu_n] = \frac{2}{\alpha'} \oint_{\mathcal{C}} \frac{dw}{2\pi i} \operatorname{Res}_{z\to w} \left[(i z^m \partial X^\mu(z)) (i w^n \partial X^\nu(w)) \right]. \tag{18}$$

We recall the familiar OPE

$$\partial X^\mu(z) \partial X^\nu(w) = -\frac{\alpha'}{2} \frac{\eta^{\mu\nu}}{(z-w)^2} + \cdots, \tag{19}$$

and where we will need to Taylor expand

$$z^m = w^m + m w^{m-1}(z-w) + \cdots \tag{20}$$

in order to evaluate the residue as $z \to w$. Using these results in (18), we find

$$\begin{aligned}
[\alpha^\mu_m, \alpha^\nu_n] &= \frac{2i^2}{\alpha'} \oint_{\mathcal{C}} \frac{dw}{2\pi i} \operatorname{Res}_{z\to w} \left[\left(w^m + m w^{m-1}(z-w) + \cdots \right) w^n \left(-\frac{\alpha'}{2} \frac{\eta^{\mu\nu}}{(z-w)^2} + \cdots \right) \right] \\
&= \oint_{\mathcal{C}} \frac{dw}{2\pi i} \operatorname{Res}_{z\to w} \left[\frac{\eta^{\mu\nu} m w^{m-1} w^n}{z-w} + \cdots \right] \\
&= \oint_{\mathcal{C}} \frac{dw}{2\pi i} m w^{n+m-1} \eta^{\mu\nu}.
\end{aligned} \tag{21}$$

The contour integral in the final line of (21) only picks up a pole if $w^{n+m-1} = w^{-1}$, which occurs when $n = -m$, in which case the entire expression is simply $m\eta^{\mu\nu}$; otherwise it vanishes. We can write this as

$$[\alpha^\mu_m, \alpha^\nu_n] = m \eta^{\mu\nu} \delta_{m,-n}, \tag{22}$$

which is the first desired commutator (2.7.5a); the calculation for the tilded oscillators $\tilde{\alpha}^\mu_n$ is identical.

Next we would like to compute $[x^\mu, p^\nu]$ to verify the commutator in equation (2.7.5b) of Polchinski. To do this, we would like to express the center-of-mass coordinate x^μ for the string, and its conjugate momentum p^ν, as currents so that we can apply equation (15). But recall x^μ comes from the mode expansion of $X^\mu(z,\bar{z})$ as given by equation (2.7.4) of Polchinski:

$$X^\mu(z,\bar{z}) = x^\mu - \frac{i\alpha'}{2}p^\mu \log\left(|z|^2\right) + i\sqrt{\frac{\alpha'}{2}}\sum_{m\neq 0}\frac{1}{m}\left(\frac{\alpha_m^\mu}{z^m} + \frac{\tilde{\alpha}_m^\mu}{\bar{z}^m}\right), \tag{23}$$

which can be inverted to give

$$x^\mu = \oint \frac{dz}{2\pi i}\underbrace{\frac{X^\mu(z,\bar{z})}{z}}_{\equiv j_x^\mu}. \tag{24}$$

Meanwhile, p^μ is defined to be proportional to the zero oscillator: $p^\mu = \sqrt{\frac{2}{\alpha'}}\alpha_0^\mu$. This can be extracted from the mode expansion for $\partial X^\mu(z,\bar{z})$ as

$$p^\mu = \oint \frac{dz}{2\pi i}\underbrace{\frac{2i}{\alpha'}\partial X^\mu}_{\equiv j_p^\mu}. \tag{25}$$

Hence, applying equation (15), we find

$$[x^\mu, p^\nu] = \oint \frac{dw}{2\pi i}\mathrm{Res}_{z\to w}\left[\frac{2i}{\alpha'}\frac{X^\mu(z)\partial X^\nu(w)}{z}\right]. \tag{26}$$

Using the OPE

$$X^\mu(z)\partial_w X^\nu(w) = \partial_w\left(-\frac{\alpha'}{2}\eta^{\mu\nu}\log(z-w)\right), \tag{27}$$

we see that (26) becomes

$$\begin{aligned}
[x^\mu, p^\nu] &= \oint \frac{dw}{2\pi i}\mathrm{Res}_{z\to w}\left[\frac{2i}{\alpha'}\partial_w\left(-\frac{\alpha'}{2}\eta^{\mu\nu}\log(z-w)\right)\right] \\
&= \oint \frac{dw}{2\pi}\mathrm{Res}_{z\to w}\left[\eta^{\mu\nu}\frac{\frac{1}{w}+\cdots}{(z-w)}\right] \\
&= i\eta^{\mu\nu}.
\end{aligned} \tag{28}$$

In the second step of (28), we have expanded $\frac{1}{z} = \frac{1}{w} + \left(-\frac{1}{w^2}\right)(z-w) + \cdots$ about $z = w$, but the higher-order terms do not contribute to the residue; in the third step, we have found the residue of $\frac{1}{w}$ from the expression in brackets and integrated the result using Cauchy's theorem. This gives the desired result.

Finally, we want to find the anti-commutator of ghost modes $\{b_m, c_n\}$ using this method. Thankfully, the analogue of (15) with the commutator replaced by an anti-commutator holds in this case. So we are interested in computing

$$\{Q_1, Q_2\} = \oint_{\mathcal{C}}\frac{dw}{2\pi i}\mathrm{Res}_{z\to w}\, j_1(z)j_2(w), \tag{29}$$

where now the currents j_i should be extracted from the expansions (2.7.16) in Polchinski, namely $b(z) = \sum_{m=-\infty}^{\infty}\frac{b_m}{z^{m+\lambda}}$ and $c(z) = \sum_{m=-\infty}^{\infty}\frac{c_m}{z^{m+1-\lambda}}$, which gives

$$\begin{aligned}
b_m &= \oint \frac{dz}{2\pi i}z^{m+\lambda-1}b(z), \\
c_m &= \oint \frac{dz}{2\pi i}z^{m-\lambda}c(z).
\end{aligned} \tag{30}$$

Keeping track of the factor of i as before, we define the currents

$$\begin{aligned}
j_m^{(b)} &= z^{m+\lambda-1}b(z), \\
j_m^{(c)} &= z^{m-\lambda}c(z).
\end{aligned} \tag{31}$$

Then equation (29) yields

$$\{b_m, c_n\} = \oint_{\mathcal{C}} \frac{dw}{2\pi i} \operatorname{Res}_{z\to w} \left[\left(z^{m+\lambda-1} b(z) \right) \left(w^{n-\lambda} c(w) \right) \right], \tag{32}$$

We recall the ghost OPE $b(z)c(w) = \frac{1}{z-w} + \cdots$, and this time we need not even Taylor expand the factor of $z^{m+\lambda-1}$ since the OPE truncates at order $(z-w)^{-1}$, so we find

$$\begin{aligned} \{b_m, c_n\} &= \oint_{\mathcal{C}} \frac{dw}{2\pi i} \operatorname{Res}_{z\to w} \left[\left(w^{m+\lambda-1} + \cdots \right) w^{n-\lambda} \frac{1}{z-w} \right], \\ &= \oint_{\mathcal{C}} \frac{dw}{2\pi i} \operatorname{Res}_{z\to w} \left[w^{m+n-1} \frac{1}{z-w} \right] \\ &= \oint_{\mathcal{C}} \frac{dw}{2\pi i} w^{m+n-1} \\ &= \delta_{m,-n}. \end{aligned} \tag{33}$$

This is the expected ghost anti-commutator.

Exercise 3. *Polchinski 2.13* 10 points

(a) Show that

$$:b(z)c(z'): - \circ\!\!\circ\, b(z)c(z') \,\circ\!\!\circ = \frac{(z/z')^{1-\lambda}-1}{z-z'} \tag{34}$$

by the method of equation (2.7.11).

(b) Use this to determine the ordering constant in N^g, equation (2.7.22). You also need the conformal transformation (2.8.14).

(c) Show that one obtains the same value for N^g by a heuristic treatment of the ordering similar to that in section 2.9.

Solution 3.

(a) We will begin by considering the product $b(z)c(w)$, expanding in fermionic creation and annihilation operators, and then using the anticommutation relations to move all of the annihilation operators to the right; this will give a relationship between the un-normal-ordered product and the creation-annihilation-normal-ordered product. From here we will show (34) by relating the un-normal-ordered product to the conformal-normal-ordered product; the latter simply subtracts off the divergent expectation value as the insertion points collide.

The ghost fields are expanded in modes as

$$\begin{aligned} b(z) &= \sum_{m=-\infty}^{\infty} \frac{b_m}{z^{m+\lambda}} \\ c(z) &= \sum_{m=-\infty}^{\infty} \frac{c_m}{z^{m+1-\lambda}}. \end{aligned} \tag{35}$$

We remember that there is a small subtlety about zero modes: b_0 is a lowering operator, but c_0 is a raising operator. Thus all of the modes b_m with $m \geq 0$ must be moved to the right of modes c_n with $n \geq 0$.

Thus the un-normal-ordered product has the mode expansion

$$b(z)c(w) = \sum_{m,n=-\infty}^{\infty} \frac{b_m c_n}{z^{m+\lambda} w^{n+1-\lambda}}. \tag{36}$$

We focus on the terms that are *not* normal ordered, i.e. those that have an annihilation operator to the left of a creation operator. These are

$$\begin{aligned} (\text{non-normal-ordered terms}) &= \sum_{m=0}^{\infty} \sum_{n=-\infty}^{0} \frac{b_m c_n}{z^{m+\lambda} w^{n+1-\lambda}} \\ &= \sum_{m=0}^{\infty} \sum_{n=-\infty}^{0} \frac{-c_n b_m + \delta_{m,-n}}{z^{m+\lambda} w^{n+1-\lambda}}, \end{aligned} \tag{37}$$

where in the second line we have used the anti-commutator (33) which we derived in the previous problem. The term involving the Kronecker delta is

$$\begin{aligned} \sum_{m=0}^{\infty} \sum_{n=-\infty}^{0} \frac{\delta_{m,-n}}{z^{m+\lambda} w^{n+1-\lambda}} &= \sum_{m=0}^{\infty} \frac{1}{z^{m+\lambda} w^{-n+1-\lambda}} \\ &= \frac{1}{w}\left(\frac{w}{z}\right)^{\lambda} \sum_{n=0}^{\infty} \left(\frac{w}{z}\right)^{n} \end{aligned} \tag{38}$$

The sum in the last line of (38) is geometric, giving $\frac{1}{1-\frac{w}{z}}$, so we find that

$$b(z)c(w) = \circ\!\!\circ\, b(z)c(w) \,\circ\!\!\circ + \left(\frac{z}{w}\right)^{1-\lambda} \frac{1}{z-w}. \tag{39}$$

On the other hand, the usual prescription for conformal normal ordering is simply to subtract off the expectation value as the insertion points collide, or

$$\begin{aligned} :b(z)c(w): &= b(z)c(w) - \lim_{z\to w}\langle b(z)c(w)\rangle \\ &= b(z)c(w) - \frac{1}{z-w}. \end{aligned} \tag{40}$$

Comparing equation (39) to equation (40), we find the desired relationship between the normal-ordering prescriptions:

$$:b(z)c(w): - \,\circ\!\!\circ\, b(z)c(w) \,\circ\!\!\circ\, = \frac{\left(\frac{z}{w}\right)^{1-\lambda} - 1}{z-w}. \tag{41}$$

(b) We wish to find the normal-ordering constant in the charge given by Polchinski's equation (2.7.22),

$$N^g = \int_0^{2\pi} \frac{dw}{2\pi i}\,(:bc:)\,, \tag{42}$$

where the normal-ordered product $:bc:\equiv j$ is the ghost number current. N^g is defined on the cylinder, which can be mapped to the plane by a conformal transformation (as we did when discussing radial quantization); the charge on the plane is related to that on the cylinder by equation (2.8.14),

$$Q^g = N^g + \lambda - \frac{1}{2}. \tag{43}$$

On the plane, the charge Q^g is defined by a contour integral rather than an ordinary integral as in (42), which allows us to relate it to the creation-annihilation-normal-ordered product and evaluate the difference using Cauchy's theorem. Explicitly,

$$\begin{aligned} N^g &= Q^g - \lambda + \frac{1}{2} \\ &= \left[-\oint \frac{dz}{2\pi i}\,(:bc:)\right] - \lambda + \frac{1}{2} \\ &= \left[-\oint \frac{dz}{2\pi i}\left((\circ\!\!\circ\, bc \,\circ\!\!\circ) + \frac{\left(\frac{z}{w}\right)^{1-\lambda} - 1}{z-w}\right)\right] - \lambda + \frac{1}{2}. \end{aligned} \tag{44}$$

To evaluate the second term in the integral of (44), we need the residue

$$\begin{aligned} \mathrm{Res}_{z\to w}\left[\frac{\left(\frac{z}{w}\right)^{1-\lambda} - 1}{z-w}\right] &= \mathrm{Res}_{z\to w}\left[\frac{\left(1^{\lambda-1} + (1-\lambda)\frac{1}{w} + \cdots\right) - 1}{z-w}\right] \\ &= 1-\lambda. \end{aligned} \tag{45}$$

Overall, then, we find

$$N^g = -\oint \frac{dz}{2\pi i}\,\circ\!\!\circ\, bc \,\circ\!\!\circ\, - \frac{1}{2}. \tag{46}$$

This gives the normal-ordering constant of $-\frac{1}{2}$, in agreement with Polchinski's equation (2.7.22).

(c) The heuristic for finding the normal-ordering constant, as described in Polchinski's section 2.9, is

1. Add the zero-point energies $\frac{1}{2}\omega$ for each bosonic mode and $-\frac{1}{2}\omega$ for each fermionic (anticommuting) mode.
2. One encounters divergent sums of the form $\sum_{n=1}^{\infty}(n-\theta)$, the θ arising when one considers nontrivial periodicity conditions. Define

$$\sum_{n=1}^{\infty}(n-\theta) = \frac{1}{24} - \frac{1}{8}(2\theta-1)^2 \,. \tag{47}$$

This is the value one obtains as in equation (1.3.32) by regulating and discarding the quadratically divergent part.

3. The above gives the normal-ordering constant for the w-frame generator T_0, equation (2.6.8). For L_0 we must add the nontensor correction $\frac{1}{24}c$.

In this case, we will work directly with the ghost number

$$N^g = -\sum_{m=-\infty}^{\infty} b_m c_{-m}, \tag{48}$$

where "adding the zero-point energies" corresponds to adding the commutators needed to move all annihilation operators to the right. This gives

$$N^g = -\sum_{m=-\infty}^{\infty} :b_m c_{-m}: + \sum_{m=0}^{\infty} 1. \tag{49}$$

Using zeta function regularization, one has

$$\sum_{m=0}^{\infty} 1 = \zeta(0) = -\frac{1}{2}. \tag{50}$$

Including the minus sign, this gives a normal-ordering constant of $-\frac{1}{2}$ as before.

Exercise 4. *Polchinski 3.13* 10 points

Consider a spacetime with d flat dimensions and 3 dimensions in the shape of a 3-sphere. Let H be proportional to the completely antisymmetric tensor on the 3-sphere and let the dilaton be constant. Using the form (3.7.14) for the equations of motion, show that there are solutions with $d+3 \neq 26$. These solutions are outside the range of validity of equation (3.7.14), but we will see in chapter 15 that there are exact solutions of this form, though with H limited to certain quantized values.

Solution 4.

In this problem, we let $D = d + 3$, so that there are d Minkowski directions and 3 directions compactified on an S^3, which we take to have radius R.

I will use upper-case Latin indices ($M,\ N = 0, \cdots, D-1$) to run over the full D dimensions, lower-case Latin indices to label directions on the three-sphere ($i, j = 1, 2, 3$), and lower-case Greek indices to label the d Minkowski directions ($\mu, \nu = 0, 4, \cdots, D-1$).

We begin with Polchinski's equations (3.7.14) for the beta functions:

$$\begin{aligned}
\beta^G_{MN} &= \alpha' R_{MN} + 2\alpha' \nabla_M \nabla_N \Phi - \frac{\alpha'}{4} H_{MLP} H_N{}^{LP} + \mathcal{O}(\alpha'^2), \\
\beta^B_{MN} &= -\frac{\alpha'}{2} \nabla^P H_{PMN} + \alpha' \left(\nabla^R \Phi\right) H_{RMN} + \mathcal{O}\left(\alpha'^2\right), \\
\beta^\Phi_{MN} &= \frac{D-26}{6} - \frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla_R \Phi \nabla^R \Phi - \frac{\alpha'}{24} H_{MNL} H^{MNL} + \mathcal{O}\left(\alpha'^2\right).
\end{aligned} \tag{51}$$

We wish to show that there exist field configurations for which the three functions (51) vanish, and such that the dilaton is constant ($\Phi(X^\mu) = \Phi_0$), and the Kalb-Ramond field strength is proportional to the volume form on the sphere. In our notation, the latter condition reads

$$\begin{aligned}
H_{ijk} &= H_0 \epsilon_{ijk}, \\
H_{i\mu\nu} = H_{ij\mu} &= H_{\mu\nu\rho} = 0.
\end{aligned} \tag{52}$$

To be explicit, I will write ϵ_{ijk} for the tensor version of the Levi-Civita symbol (which contains a factor of $\sqrt{g}$) and ε_{ijk} for the tensor density version:

$$\epsilon_{ijk} = \left\{ \begin{array}{ll} \sqrt{\det\left(g_{mn}\right)}, & \text{if } i,j,k \text{ is an even permutation of } 1,2,3 \\ -\sqrt{\det\left(g_{mn}\right)}, & \text{if } i,j,k \text{ is an odd permutation of } 1,2,3 \\ 0, & \text{otherwise} \end{array} \right\} = \sqrt{\det\left(g_{mn}\right)} \varepsilon_{ijk}. \tag{53}$$

To show that such solutions exist, we will plug in our ansatzes for Φ and H_{MNP} into the β function equations (51) and show that they can be made to vanish. One finds that the required conditions, to order α', are

$$\begin{aligned}
0 &= \alpha' R_{MN} - \frac{\alpha'}{4} H_{MLP} H_N{}^{LP}, \\
0 &= \nabla^P H_{PMN}, \\
0 &= \frac{D-26}{6} - \frac{\alpha'}{24} H_{MNL} H^{MNL}.
\end{aligned} \tag{54}$$

Note that our spacetime is $S^3 \times M^d$, where M^d is d-dimensional Minkowski space, so the Ricci tensor satisfies $R_{\mu\nu} = 0 = R_{\mu i}$ when either index is Greek. Likewise, H_{PMN} vanishes when any index is Greek. Thus the only non-trivial equations coming from (54) are

$$\begin{aligned}
R_{ij} &= \frac{H_0^2}{4} \epsilon_{imp} \epsilon_j{}^{mp}, \\
0 &= \nabla^p \left(H_0 \epsilon_{pmn}\right), \\
\frac{D-26}{6} &= \frac{\alpha' H_0^2}{24} \epsilon_{ijk} \epsilon^{ijk}.
\end{aligned} \tag{55}$$

The second equation in (55) is automatically satisfied, since $\nabla^p \sqrt{\det(g_{ij})} = 0$ by the metric compatibility assumption.T

Next, since ϵ_{ijk} with upstairs indices is just

$$\epsilon^{ijk} = \left\{ \begin{array}{lr} \frac{1}{\sqrt{\det(g_{mn})}}, & \text{if } i,j,k \text{ is an even permutation of } 1,2,3 \\ -\frac{1}{\sqrt{\det(g_{mn})}}, & \text{if } i,j,k \text{ is an odd permutation of } 1,2,3 \\ 0, & \text{otherwise} \end{array} \right\} = \frac{1}{\sqrt{\det\left(g_{mn}\right)}} \varepsilon^{ijk}, \tag{56}$$

the product $\epsilon_{ijk}\epsilon^{ijk} = \varepsilon_{ijk}\varepsilon^{ijk} = 6$, just as for the tensor density version of the Levi-Civita symbol. Hence the third equation of (55) gives a constraint on the total spacetime dimension D, namely

$$D = d + 3 = 26 + \frac{\alpha' H_0^2}{4}. \tag{57}$$

Said differently: requiring that the spacetime dimension be an integer, the condition (57) requires that the H flux be quantized.

Our final task is to show that the first equation in (55) can be satisfied. I must confess that I lack the patience for computing curvatures by hand, so I will use my handy Mathematica package.

```
In[1]:= << SUGRA`
        SUGRA has been loaded.

In[7]:= S3Coords = {ψ, θ, ϕ};
        ds2 = R^2 (dψ^2 + Sin[ψ]^2 (dθ^2 + Sin[θ]^2 dϕ^2));
        Gdd = Metric[ds2, S3Coords];
        Guu = IMetric[Gdd];
        Rdddd = Normal@goodRiemann[Gdd, S3Coords];

In[14]:= Rdd = TensorMath[Rdddd[[-i, -j, -k, -m]] Guu[[i, k]]];
         S[SAR[Symmetrize[Rdd, Symmetric[All]]]]

Out[15]= {{1, 1} → 2, {2, 2} → 2 Sin[ψ]^2, {3, 3} → 2 Sin[θ]^2 Sin[ψ]^2, {_, _} → 0}

In[16]:= S[SAR[Symmetrize[Gdd, Symmetric[All]]]]

Out[16]= {{1, 1} → R^2, {2, 2} → R^2 Sin[ψ]^2, {3, 3} → R^2 Sin[θ]^2 Sin[ψ]^2, {_, _} → 0}
```

The Ricci tensor has only three components, namely $R_{11} = 2$, $R_{22} = 2\sin^2(\psi)$, and $R_{33} = 2\sin^2(\theta)\sin^2(\psi)$. Incidentally, these are the same as the metric components re-scaled by $\frac{2}{R^2}$, so we have found

$$R_{ij} = \frac{2}{R^2} g_{ij}. \tag{58}$$

But this is just what we want, because the right side of the first equation in (55) is

$$\frac{H_0^2}{4} \epsilon_{imp} \epsilon_j{}^{mp} = \frac{H_0^2}{4} g_{jk} \underbrace{\epsilon_{imp}\epsilon^{kmp}}_{2\delta_i{}^k} = \frac{H_0^2}{2} g_{ij}, \tag{59}$$

so the beta function for the metric will vanish provided that

$$H_0^2 = \frac{4}{R^2}. \tag{60}$$

Thus we have found that the proposed field configuration for g_{MN}, H_{MNP}, and Φ satisfies the vanishing conditions for all three beta functions if the two conditions (57) and (60) are met, as desired.

Exercise 5. *Polchinski 5.1* 10 points

Consider the sum (5.1.1), but now over all particle paths beginning and ending at given points in spacetime. In this case fixing the tetrad leaves no coordinate freedom.

(a) Derive the analog of the gauge-fixed path integral (5.3.9).

(b) Reduce the ghost path integral to determinants as in equation (5.3.18).

(c) Reduce the X^μ path integral to determinants (as will be done in section 6.2 for the string).

(d) Evaluate the finite and functional determinants and show that the result is the scalar propagator for a particle of mass m.

Solution 5.

(a) We begin with the point particle path integral

$$\int_{X_i^\mu}^{X_f^\mu} \mathcal{D}e\, \mathcal{D}X^\mu \, \exp\left[-\frac{1}{2} \int d\tau \, \left(e^{-1} \dot{X}^\mu \dot{X}_\mu + em^2 \right) \right], \tag{61}$$

where the endpoints mean that we consider all paths beginning at an initial point X_i^μ and ending at a final point X_f^μ. Our goal is to write this path integral in a way similar to Polchinski's equation (5.3.9), which has fixed the diffeomorphism invariance by introducing ghost fields b and c.

In the point particle case, our gauge freedom comes from worldline diffeomorphisms for the einbein field e. Under an infinitesimal reparameterization $\tau \to \tau - \xi(\tau)$, the einbein transforms as $\delta e = \partial_\tau (\xi e)$, which leaves the action invariant. Speaking loosely, then, our path integral (61) integrates over "too many" field configurations, since we include many equivalent choices which are related by reparameterization.

To gauge fix, we would like to "cut the gauge orbits just once" – that is, we want to choose a single representative einbein e for each class of equivalent einbeins, and then integrate only over the inequivalent representatives. For instance, we could choose the representative einbeins to be constant. But note that, once we have chosen a prescription for picking representatives (e.g. a constant einbein), the length of the path of the particle is

$$L = \int_{\tau_i}^{\tau_f} d\tau \, e = e\,(\tau_f - \tau_i)\,, \tag{62}$$

so we must have $e = \frac{L}{\tau_f - \tau_i}$. For simplicity, let's choose $\tau_i = 0$ and $\tau_f = 1$, so the einbein is fixed to the constant value $e = L$ for a path of length L. This length is a modulus that we will need to integrate over.

Now we wish to insert a delta function in the path integral which fixes the einbein to the constant value L chosen above. An expression of the form $\int \mathcal{D}\xi \, \delta\left(e - e_L^\xi\right)$ will not be quite 1 due to the analogue of the delta function rule $\int dx\, \delta(f(x)) = \frac{1}{|f'(x_0)|}$, but it will give an expression which is the inverse of what we call the Fadeev-Popov determinant:

$$\Delta_{FP}^{-1}(e) = \int_0^\infty dL \, \int \mathcal{D}\xi \, \delta\left(e - e_L^\xi\right), \tag{63}$$

where the somewhat clumsy notation $\delta\left(e - e_L^\xi\right)$ means "a delta function which fires only when the two einbeins e and e_L^ξ, the latter of which is related to the constant einbein $e = L$ through a reparameterization by some function ξ, are equal."

I now follow the procedure outlined in section 5.1.2 of Tong's notes, where we compute Δ_{FP} by considering gauge transformations (here reparameterizations) which are close to the identity. This is sensible, since the delta function will only fire when its argument vanishes, so it should be enough to consider only cases where the argument is close to zero.

We have already observed that, under a reparameterization by some infinitesimal ξ, the einbein transforms as $\delta e = \partial_\tau (\xi e)$. We should also allow the total length of the path to wiggle, $L \to L + \delta L$, to consider the most general fluctuation of the einbein. Thus the delta function can be written as

$$\delta\left(e - e^{\xi}_{L+\delta L}\right) = \delta\left(\partial_\tau\left(\xi e\right) + \frac{\partial e_L}{\partial L}\delta L\right). \tag{64}$$

Next, as in Tong, we would like to express this delta function as an integral, just as we could write $\delta(x) = \int dp\, e^{2\pi i p x}$ for an ordinary delta function. In this case, we have a delta functional, so the analogous formula must involve a functional integration over some worldline field which we will call β, and the argument of the exponential will be a worldline integral of β against the stuff inside the delta function.

To be diff-invariant, note that this worldline integral in the exponential must come with a measure of $d\tau$ rather than the usual $e\, d\tau$. This is because the integrand is of the form $\beta\, \delta e$, and we will choose the convention that β is a scalar under worldline diffeomorphisms. Since δe transforms like e, the product $d\tau\, \delta e$ is already a scalar. Using this measure, we find

$$\delta\left(e - e^{\xi}_{L}\right) = \int \mathcal{D}\beta\, \exp\left(2\pi i \int_0^1 d\tau\, \beta\left(\partial_\tau\left(\xi e\right) + \frac{\partial e_L}{\partial L}\delta L\right)\right). \tag{65}$$

Actually, it will be somewhat more convenient to re-scale the function ξ appearing in (65). Recall that we have chosen ξ to parameterize infinitesimal changes $\tau \to \tau' = \tau - \xi(\tau)$, but the combination $e\xi$ is what appears in our integral. We may as well define a new function $\gamma = e\xi$, and integrate over infinitesimal γ instead. All in all, our inverse determinant then becomes

$$\Delta_{FP}^{-1}(e) = \int_0^\infty dL \int \mathcal{D}\gamma\, \mathcal{D}\beta\, d\delta L\, \exp\left(2\pi i \int_0^1 d\tau\, \beta\left(\partial_\tau\gamma + \frac{\partial e_L}{\partial L}\delta L\right)\right). \tag{66}$$

Now we use the usual trick of replacing commuting variables with anticommuting ones. Let's recall (a very non-rigorous summary of) the reasoning: our integral (66) is Gaussian, so it is computing the inverse determinant of some operator, by analogy with the usual formula

$$\int e^{-\frac{1}{2}x_i A_{ij} x_j}\, d^n x = \sqrt{\frac{(2\pi)^n}{\det(A)}}. \tag{67}$$

This is why we have used the suggestive notation $\Delta_{FP}^{-1}(e)$. But the corresponding formula for an integral over *anticommuting* variables is

$$\int e^{-\theta^T A \eta}\, d\theta\, d\eta = \det(A). \tag{68}$$

So replacing all bosonic fields with Grassmann-valued fields should, morally, invert the determinant[1], up to some constants. Let's be careful about the constants: because the integral (66) has the wrong sign (it comes with a plus sign, whereas (67) and (68) have minus signs), we will get a relative factor of $\frac{1}{i}$ from the square root of the determinant of A. Also, the prefactor of 2π is a factor of 4π larger than the prefactor of $\frac{1}{2}$ in (67). Overall, I think the prefactor of $2\pi i$ is therefore replaced by a prefactor of $\frac{1}{4\pi}$.

With that said, we shall replace γ by c, β by b, and δL by λ, and correct the constant prefactor. This gives an expression for the un-inverted Fadeev-Popov determinant,

$$\Delta_{FP}(e) = \int_0^\infty dL \int \mathcal{D}c\, \mathcal{D}b\, d\lambda\, \exp\left(\frac{1}{4\pi}\int_0^1 d\tau\, b\left(\partial_\tau c + \frac{\partial e_L}{\partial L}\lambda\right)\right). \tag{69}$$

The integral over λ is easy, so let's do it now. For anticommuting θ, we have $\int e^{a\theta}\, d\theta = \int (1 + a\theta)\, d\theta = a$, so the integral over λ pulls down a multiplicative factor of the stuff multiplying λ in the exponential.

Doing this, we find the gauge-fixed path integral

$$\int_0^\infty dL \int_{X_i^\mu}^{X_f^\mu} \mathcal{D}X \int \mathcal{D}c\, \mathcal{D}b\left(\frac{1}{4\pi}\int_0^1 d\tau\, b\frac{\partial e_L}{\partial L}\right)\exp\left(-\frac{1}{2}\int d\tau\, \left(e^{-1}\dot{X}^\mu \dot{X}_\mu + e m^2\right) + \frac{1}{4\pi}\int d\tau\, b\,\partial_\tau c\right). \tag{70}$$

[1] I don't actually understand why the inversion seems insensitive to the fact that the *square root* of the determinant appears in the bosonic case, but the determinant to the first power appears in the anticommuting case.

(b) We wish to trade our integral over the ghost fields b and c for a product of determinants which we can evaluate. To do this, we will follow the strategy Polchinski uses on pages 157 and 158. Begin by expanding the ghost fields in complete sets as

$$\begin{aligned} c(\tau) &= \sum_J c_J \mathcal{C}_J(\tau), \\ b(\tau) &= \sum_J b_J \mathcal{B}_J(\tau), \end{aligned} \tag{71}$$

A convenient choice for the basis states $\mathcal{C}_J(\tau)$ and $\mathcal{B}_J(\tau)$ is to use eigenfunctions of the Laplacian, which on the worldline is

$$\Delta = -\frac{1}{e^2}\frac{\partial^2}{\partial\tau^2}. \tag{72}$$

In a gauge where e is constant, this is simply proportional to $-\partial_\tau^2$. The eigenfunctions can be chosen as complex exponentials or sines and cosines, but the latter is more useful given our boundary conditions. Recall that the c field came from the reparameterization by some function $\xi(\tau)$ on the worldline, which is assumed to vanish at the endpoints; thus $c(\tau)$ should also be zero at the endpoints, so we choose to expand it in sines

$$c(\tau) = \sqrt{\frac{2}{L}}\sum_J c_J \sin\left(\pi J\tau\right). \tag{73}$$

The ghost integral $\int d\tau\, b\partial_\tau c$ then becomes a sum of integrals of b against cosines, so only the cosines in the $b(\tau)$ expansion will survive (and possibly a zero mode, which contributes to the prefactor in the path integral). We therefore write

$$b(\tau) = \frac{b_0}{\sqrt{L}} + \sqrt{\frac{2}{L}}\sum_{J=1}^{\infty} b_J\cos(\pi J\tau) \tag{74}$$

The ghost action then becomes

$$\begin{aligned} \frac{1}{4\pi}\int_0^1 d\tau\, b\,\partial_\tau c &= \frac{1}{4\pi}\int_0^1 d\tau\left(\frac{b_0}{\sqrt{L}} + \sqrt{\frac{2}{L}}\sum_{J=1}^{\infty} b_J\cos(\pi J\tau)\right)\frac{\partial}{\partial\tau}\left(\sqrt{\frac{2}{L}}\sum_K c_K\sin\left(\pi K\tau\right)\right) \\ &= \frac{1}{2L}\sum_{J=1}^{\infty} J b_J c_J. \end{aligned} \tag{75}$$

Meanwhile, the prefactor appearing before the path integral contributes

$$\frac{1}{4\pi}\int_0^1 d\tau\, b\frac{\partial e_L}{\partial L} = \frac{b_0}{4\pi\sqrt{L}}. \tag{76}$$

Now we can trade our path integral over b and c for ordinary integrals over each of the modes b_J and c_J, each of which is then Gaussian. More explicitly,

$$\begin{aligned} &\int \mathcal{D}b\,\mathcal{D}c\left(\frac{b_0}{4\pi\sqrt{L}}\right)\exp\left(\frac{1}{2L}\sum_J J b_J c_J\right) \\ &= \int\left(\prod_{n=0}^{\infty} b_n\right)\left(\prod_{m=1}^{\infty} c_m\right)\left(\frac{b_0}{4\pi\sqrt{L}}\right)\exp\left(\frac{1}{2L}\sum_J J b_J c_J\right). \end{aligned} \tag{77}$$

Now we cite the familiar result for Gaussian integrals over Grassmann variables: if θ and η are vectors of anti-commuting numbers and A is a matrix, then

$$\int \exp\left(-\theta^T A\eta\right) d\theta\, d\eta = \det(A). \tag{78}$$

Meanwhile, the zero-mode contribution is trivial, since $\int db_0\, b_0 = 1$ for Grassmann b_0. We conclude that the integral gives

$$\frac{1}{4\pi\sqrt{L}}\prod_{J=1}^{\infty}\frac{J}{2L}. \tag{79}$$

We will need to regulate this infinite product in part (d).

(c) For the bosonic fields, we would like to split the X^μ into the "instanton" piece which solves the classical equations of motion, plus a fluctuation, and then integrate over all fluctuations.

The classical solutions X_0^μ satisfy $\partial_\tau^2 X_0^\mu = 0$, so they are the constant-velocity paths between the endpoints X_μ^i and X_μ^f. Thus we can expand X^μ as

$$X^\mu(\tau) = \underbrace{X_i^\mu + \left(X_f^\mu - X_i^\mu\right)\tau}_{\equiv X_0^\mu} + \delta X^\mu(\tau). \tag{80}$$

The boundary conditions require $\delta X^\mu = 0$ at $\tau = 0$ and $\tau = 1$, so we should Fourier-expand it in sines:

$$\delta X^\mu(\tau) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} a_n^\mu \sin(n\pi\tau). \tag{81}$$

Now we replace the action for X^μ, in the gauge where e is constant, with the constant contribution from the classical path plus the action for the fluctuations. The latter has the form

$$\int_0^1 d\tau \left[e^{-1}\left(\delta\dot{X}^\mu\right)^2 + em^2\right] = Lm^2 + \frac{\pi^2}{L^2}\sum_{n=1}^{\infty} a_n^\mu n^2. \tag{82}$$

As before, we trade the path integral $\mathcal{D}\delta X^\mu$ for ordinary integrals over the Fourier modes a_n. The resulting integrals look like

$$\int \left(\prod_{n,\mu} da_n^\mu\right) \exp\left(-\frac{\pi^2}{L^2}\sum_j j^2 \left(a_j^\mu\right)^2\right). \tag{83}$$

To evaluate each of these, we use the version of equation (78), which gives the Gaussian integral of Grassmann numbers, that applies for commuting numbers:

$$\int d^n x \, \exp\left(-\frac{1}{2}x_i A_{ij} x_j\right) = \sqrt{\frac{(2\pi)^n}{\det(A)}}. \tag{84}$$

Our result, then, is proportional to

$$\prod_{n=1}^{\infty} \frac{\pi^2 n^2}{L^2}. \tag{85}$$

(d) Here we will need to infinite products of the form

$$\prod_{k=0}^{\infty} \frac{k^2}{L^2} \quad \text{and} \quad \prod_{k=0}^{\infty} \frac{k}{L} \tag{86}$$

which appeared (up to constant prefactors) in parts (b) and (c) above.

There are two ways to do this.

Method 1: Zeta function regularization.

See Box 1 for a brief review of how zeta function regularization works for products. In this scheme, we can split the infinite product $\prod_{k=1}^{\infty} \frac{k^a}{L^2}$ into two separate products, one coming from the numerator and one from the denominator[2]

[2] For more about the rules of manipulating zeta-regularized products, see this paper.

Box 1. A Primer on Zeta Function Regularization

The idea here is to exploit an analogy with ζ-function regularization of sums. To define a divergent sum $\sum_{n=1}^{\infty} a_n$, we might first define

$$\zeta_a(s) = \frac{1}{a_1^s} + \frac{1}{a_1^s} + \cdots , \tag{87}$$

which is defined when s has sufficiently large real part and can be analytically continued elsewhere; in particular, the value of this continuation at $s = -1$ gives the desired sum.

Likewise, since the infinite product of a bunch of numbers λ_n can be written in terms of the exponential of their sum,

$$\log\left(\prod_{n=1}^{\infty} \lambda_n\right) = \sum_{n=1}^{\infty} \log\left(\lambda_n\right), \tag{88}$$

we might again define

$$\zeta_\lambda(s) = \frac{1}{\lambda_1^s} + \frac{1}{\lambda_2^s} + \cdots . \tag{89}$$

To get logs in the expression, recall that $\frac{d}{dx}\left(a^x\right) = a^x \log a$, so

$$\zeta_\lambda'(s) = \frac{-\log\lambda_1}{\lambda_1^s} - \frac{\log\lambda_2}{\lambda_2^s} + \cdots , \qquad \text{and hence} \tag{90}$$

$$\zeta_\lambda'(0) = -\log\lambda_1 - \log\lambda_2 - \cdots . \tag{91}$$

Thus one way to define the infinite product $\lambda_1\lambda_2\cdots$ is to exponentiate (91), giving

$$\prod_{n=1}^{\infty} \lambda_n \overset{😜}{\equiv} \exp\left(-\zeta_\lambda'(0)\right). \tag{92}$$

Here, of course, the notation 😜 indicates playful lightheartedness at the seemingly silly idea of defining a divergent infinite product in terms of the derivative of a zeta function.

The first product is

$$\prod_{k=1}^{\infty} k^a = e^{-a\zeta'(0)}, \tag{93}$$

but a well-known formula gives the derivative of the Riemann zeta function at zero as

$$\zeta'(0) = -\log\left(\sqrt{2\pi}\right), \tag{94}$$

so we find that

$$\prod_{k=1}^{\infty} k^a = (2\pi)^{a/2}. \tag{95}$$

On the other hand, using the definition $\prod_{n=1}^{\infty} \lambda_n = \exp\left(-\zeta_\lambda(0)\right)$, the product of the constant $\frac{1}{L^2}$ piece is

$$\prod_{k=1}^{\infty} \frac{1}{L^2} = L^{-2\zeta(0)} = L, \tag{96}$$

since $\zeta(0) = -\frac{1}{2}$.

Thus, using zeta function regularization, we can quickly evaluate the two infinite products that appear in the ghost and X^μ path integrals:

$$\prod_{J=1}^{\infty} \frac{J}{2L} \xrightarrow{\zeta} 2\sqrt{\pi L},$$
$$\prod_{n=1}^{\infty} \frac{n^2}{L^2} \xrightarrow{\zeta} 2\pi L. \tag{97}$$

Method 2: Pauli-Villars.

This is the approach used by Polchinski in his treatment of the harmonic oscillator in appendix A.1. First note that we can write both of the infinite products (97) in terms of the determinant of the Laplacian. One has

$$\det\left(-\frac{1}{L^2}\partial_\tau^2\right) = \prod_{n=1}^{\infty} \frac{n^2}{L^2},$$
$$\sqrt{\det\left(-\frac{1}{16L^2}\partial_\tau^2\right)} = \sqrt{\prod_{n=1}^{\infty} \frac{n^2}{16L^2}} = \prod_{J=1}^{\infty} \frac{J}{4L}. \tag{98}$$

Thus it will be enough for us to regulate a quantity which looks like $\det(c\Delta)$, where $\Delta = -\frac{1}{L^2}\partial_\tau^2$, and raise the result to various powers.

We regulate the determinant by dividing by another determinant with a heavy regulator controlled by some Ω as

$$\det(c\Delta) \overset{\text{Pauli-Villars}}{=} \frac{\det(c\Delta)}{\det(c\Delta + \Omega^2)} = \prod_{k=1}^{\infty} \frac{\frac{c^2k^2\pi^2}{L^2}}{\frac{c^2k^2\pi^2}{L^2} + \Omega^2}$$
$$= \left(\prod_{k=1}^{\infty}\left(1 + \frac{\frac{\Omega^2L^2}{c^2\pi^2}}{k^2}\right)\right)^{-1}. \tag{99}$$

Here we will need a result expressing the sin function as an infinite product[3], namely

$$\frac{\sin(\pi x)}{\pi x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2}\right). \tag{100}$$

Using (100) when $x = \frac{i\Omega L}{c\pi}$, and recalling that $\sin(ix) = i\sinh(x)$, we find that

$$\left(\prod_{k=1}^{\infty}\left(1 + \frac{\frac{\Omega^2L^2}{c^2\pi^2}}{k^2}\right)\right)^{-1} = \frac{\frac{\Omega L}{c}}{\sinh(\frac{\Omega L}{c})}. \tag{101}$$

Finally, we take our regulator Ω to be large. Since $\sin(x) = \frac{e^x - e^{-x}}{2}$, for large x we may replace it by $\frac{e^x}{2}$, so that

$$\det(c\Delta) \overset{\text{Pauli-Villars}}{=} 2\frac{\Omega L}{c}\exp\left(-\frac{\Omega L}{c}\right). \tag{102}$$

How does this make contact with our results (97)? We will get $\frac{1-D}{2}$ copies of (102), coming from the ghost and X^μ integrals, which can then be absorbed into the remaining classical part of the path integral

$$Z = \int_0^\infty \frac{dL}{\sqrt{L}} e^{-S_0}\left(2\Omega L\exp(-\Omega L)\right)^{\frac{1-D}{2}}$$
$$= \int_0^\infty \frac{dL}{\sqrt{L}} \exp\left(-S_0 + \frac{1-D}{2}(-\Omega L) + \frac{1-D}{2}\log(2\Omega L)\right). \tag{103}$$

[3] This can be proved most directly using complex analysis techniques.

There are two divergences in (103) to worry about as we take $\Omega \to \infty$. The linear divergence can be canceled by including a counterterm in the Lagrangian. The remaining logarithmic divergence will produce a wavefunction renormalization that agrees with the constant prefactors that we found using the zeta function method.

Rather than continue to pursue this path, I will now return to the zeta function results (97) to complete the rest of this problem, since I find that method simpler and more powerful.

To finish the calculation, we must the appropriate constants from the zeta function regularization, and perform the remaining integral over the modulus L. We still have not computed the instanton contribution to the bosonic action, which comes from the classical path:

$$\begin{aligned} S_0 &= -\frac{1}{2}\int_0^1 d\tau\,\left(\frac{1}{L}\partial_\tau\left(X_i^\mu + \left(X_f^\mu - X_i^\mu\right)\tau\right)^2 + LM^2\right) \\ &= -\frac{1}{2}\left(\frac{\left(X_f^\mu - X_i^\mu\right)^2}{L} - LM^2\right). \end{aligned} \tag{104}$$

Let's track some constants. There is one factor of $2\sqrt{L}$ coming from the first infinite product in (97), which cancels the $\frac{1}{\sqrt{L}}$ from the prefactor in the path integral which came from b_0, and then there is a factor of $(2\pi)^{-D}$ from the second product in (97). So our integral looks like

$$Z = 2\,(2\pi)^{-D}\int_0^\infty dL\,L^{-D/2}\exp\left(-\frac{1}{2}\left(\frac{\left(X_f^\mu - X_i^\mu\right)^2}{L} - LM^2\right)\right). \tag{105}$$

To recover the usual momentum-space propagator, we will Fourier transform. Replace the distance $X_f^\mu - X_i^\mu$ by ΔX^μ; we will transform with respect to this position difference.

$$\tilde{Z}(k) = 2\,(2\pi)^{-D}\int_0^\infty dL\,L^{-D/2}\int d^D\left(\Delta X^\mu\right)\exp\left(ik_\mu\left(\Delta X\right)^\mu - \frac{1}{2}\left(\frac{\left(\Delta X^\mu\right)^2}{L} - LM^2\right)\right). \tag{106}$$

The integral is Gaussian, giving

$$\begin{aligned} \tilde{Z}(k) &= 2\int_0^\infty dL\,\exp\left(-\frac{L(k^2+m^2)}{2}\right) \\ &= \frac{1}{k^2+m^2}. \end{aligned} \tag{107}$$

This is exactly the scalar propagator, including the correct constant prefactor. Notice that the results (97) were precisely those required to cancel off the extra factors of 2 and π raised to various powers; if we did not include those results (or if we had used Pauli-Villars and not treated the logarithmic divergence correctly), then we would have arrived at a result which was some constant multiple of the correct scalar propagator.

Exercise 6. *Polchinski 6.13* 10 points

(a) Find the $PSL(2,\mathbb{C})$ transformation that takes three given points $z_{1,2,3}$ into chosen positions $\hat{z}_{1,2,3}$.

(b) Verify the Mobius invariance results (6.7.3) - (6.7.7). Show that to derive (6.7.5) it is sufficient that L_1 and $\tilde{L}_1$ annihilate the operators.

Solution 6.

(a) Begin with our starting points z_1, z_2, z_3. We are interested in finding an element $g \in PSL(2,\mathbb{C})$, can be represented as a 2×2 matrix of the form

$$g \simeq \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{108}$$

such that $a, b, c, d \in \mathbb{C}$, and $ad - bc = 1$, and where two matrices are considered identical if they differ only by reversing the signs of all elements. Elements of $PSL(2,\mathbb{C})$ have a group action on a complex number z which is described in terms of such a matrix representation as

$$g : z \mapsto \frac{az+b}{cz+d}. \tag{109}$$

We will simplify the task of finding $g : z_i \mapsto \hat{z}_i$ by breaking it into two steps: we write $g = g_2 \circ g_1$, where g_1 maps the z_i to the three points 0, 1, ∞, and where g_2 maps the three points 0, 1, ∞ to $\hat{z}_i$.

First let's find g_1. In order to send z_1 to 0, the numerator of the transformation (109) should vanish when $z = z_1$. Likewise, in order to send z_3 to ∞, the denominator should vanish when $z = z_3$. So far, our ansatz for the map is

$$z \mapsto \frac{A(z-z_1)}{B(z-z_3)}. \tag{110}$$

To pin down A and B, we want the map to send z_2 to 1, so that $\frac{A(z_2-z_1)}{B(z_2-z_3)} = 1$. This works if $A = z_2 - z_3$ and $B = z_2 - z_1$. Then our proposal for g_1 is

$$\begin{aligned} g_1 : z \mapsto & \frac{(z_2-z_3)(z-z_1)}{(z_2-z_1)(z-z_3)} \\ &= \frac{\overbrace{(z_2-z_3)}^{a}\, z + \overbrace{(-z_1)\,(z_2-z_3)}^{b}}{\underbrace{(z_2-z_1)}_{c}\, z + \underbrace{(-z_3)\,(z_2-z_1)}_{d}}. \end{aligned} \tag{111}$$

Likewise, the map which sends the three points $\hat{z}_i$ to $0, 1, \infty$ – and is hence the *inverse* of the element g_2 we wish to compute next – is simply

$$g_2^{-1} : z \mapsto \frac{(\hat{z}_2-\hat{z}_3)z + (-\hat{z}_1)\,(\hat{z}_2-\hat{z}_3)}{(\hat{z}_2-\hat{z}_1)z + (-\hat{z}_3)\,(\hat{z}_2-\hat{z}_1)}. \tag{112}$$

Happily, it is easy to invert a $PSL(2,\mathbb{C})$ transformation which is represented by a matrix, since we may simply take the inverse $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]^{-1} = \frac{1}{ad-bc}\left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$. Then we see g_2 can be written as

$$g_2 : z \mapsto \frac{(-\hat{z}_3)\,(\hat{z}_2-\hat{z}_1)\,z - (-\hat{z}_1)\,(\hat{z}_2-\hat{z}_3)}{-(\hat{z}_2-\hat{z}_3)z + (\hat{z}_2-\hat{z}_3)}. \tag{113}$$

The full transformation we wish to write down, then, is obtained by composing $g_2 \circ g_1$, or writing the extremely unwieldly expression

$$(g_2 \circ g_1) : z \longmapsto \frac{(-\hat{z}_3)\,(\hat{z}_2-\hat{z}_1)\,\frac{(z_2-z_3)z+(-z_1)(z_2-z_3)}{(z_2-z_1)z+(-z_3)(z_2-z_1)} - (-\hat{z}_1)\,(\hat{z}_2-\hat{z}_3)}{-(\hat{z}_2-\hat{z}_3)\frac{(z_2-z_3)z+(-z_1)(z_2-z_3)}{(z_2-z_1)z+(-z_3)(z_2-z_1)} + (\hat{z}_2-\hat{z}_3)} \tag{114}$$

Using Mathematica, I find that we can write equation (114) in the form $z \mapsto \frac{az+b}{cz+d}$, where a, b, c, d are given by

$$\begin{aligned}
a &= \hat{z}_1\hat{z}_2(z_1 - z_2) + \hat{z}_2\hat{z}_3(z_2 - z_3) + \hat{z}_1\hat{z}_3(z_3 - z_1), \\
b &= \hat{z}_1\hat{z}_3 z_2(z_1 - z_3) + \hat{z}_1\hat{z}_2 z_3(z_2 - z_1) + \hat{z}_2\hat{z}_3 z_1(z_3 - z_2), \\
c &= \hat{z}_3(z_2 - z_1) + \hat{z}_2(z_1 - z_3) + \hat{z}_1(z_3 - z_2), \\
d &= \hat{z}_1 z_1(z_2 - z_3) + \hat{z}_3 z_3(z_1 - z_2) + \hat{z}_2 z_2(z_3 - z_1).
\end{aligned}$$

This gives the desired map which sends (z_1, z_2, z_3) to $(\hat{z}_1, \hat{z}_2, \hat{z}_3)$.

(b) Let's begin with Polchinski's equation (6.7.3), which gives the behavior of a primary operator $\mathscr{A}_i$ with weights $(h_i, \tilde{h}_i)$ under the transformation $z \mapsto z' = \gamma z$.

By definition, an operator $\mathscr{A}_i$ of weight $(h_i, \tilde{h}_i)$ will transform under an arbitrary map $z \to w(z)$, $\bar{z} \to \bar{w}(\bar{z})$ as

$$\mathscr{A}_i \to \mathscr{A}_i' = \left(\frac{\partial w}{\partial z}\right)^{h_i} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{\tilde{h}_i}. \tag{115}$$

Applying this to the case of a re-scaling $z \to \gamma z$, $\bar{z} \to \gamma \bar{z}$ and enclosing the result in an expectation value, we have

$$\langle \mathscr{A}_i(0,0)\rangle S^2 \mapsto \langle \mathscr{A}_i'(0,0)\rangle_{S^2} = \gamma^{-h_i}\bar{\gamma}^{-\tilde{h}_i}\langle \mathscr{A}_i(0,0)\rangle_{S^2}. \tag{116}$$

On the other hand, the map $z \to \gamma z$ is a Mobius transformation[4] with $\beta = \gamma = 0$, so the expectation values $\langle \mathscr{A}_i(0,0)\rangle S^2$ and $\langle \mathscr{A}_i'(0,0)\rangle S^2$ in (116) must be equal. This established Polchinski's equation (6.7.3).

Next consider Polchinski's equation (6.7.4). We can make a Mobius transformation with $\alpha = 1$, $\beta = -z_2$, $\gamma = 0$, and $\delta = 1$, which sends the point z_1 to $z_1 - z_2$ and the point z_2 to zero. By invariance under Mobius transformations, then, one has

$$\langle \mathscr{A}_i(z_1, \bar{z}_1)\mathscr{A}_j(z_2, \bar{z}_2)\rangle_{S^2} = \langle \mathscr{A}_i(z_1 - z_2, \bar{z}_1 - \bar{z}_2)\mathscr{A}_j(0,0)\rangle_{S^2} \tag{117}$$

Now we can use the same argument that we used in the verification of equation (6.7.3) above – in particular, we re-scale the coordinates by $z \to z' = \frac{z}{z_1 - z_2}$. This can also be accomplished by a Mobius transformation, so the correlation function is still invariant, and using the known transformation properties of operators with weights $(h_i, \tilde{h}_i)$ and $(h_j, \tilde{h}_j)$, one finds

$$\langle \mathscr{A}_i(z_1, \bar{z}_1)\mathscr{A}_j(z_2, \bar{z}_2)\rangle_{S^2} = (z_1 - z_2)^{-h_i - h_j} (\bar{z}_1 - \bar{z}_2)^{-\tilde{h}_i - \tilde{h}_j} \langle \mathscr{A}_i(1,1)\mathscr{A}_j(0,0)\rangle_{S^2}. \tag{118}$$

This is the expected result, Polchinski's (6.7.4).

Next we turn to (6.7.5). Following the hint in the text, consider an infinitesimal conformal transformation

$$z \to z' = z + \epsilon(z - z_1)(z - z_2). \tag{119}$$

This transformation can be reproduced by an infinitesimal Mobius transformation, plus a translation. Indeed, consider $z \mapsto z' = \frac{az+b}{cz+d}$ with $a = 1$, $b = -z_1$, $c = -\epsilon$, and $d = 1 + \epsilon z_2$. Then

$$\begin{aligned}
z' = \frac{az+b}{cz+d} &= \frac{z - z_1}{1 - \epsilon(cz - z_2)} \\
&= (z - z_1)\left(1 + \epsilon(z - z_2) + \mathcal{O}(\epsilon^2)\right) \\
&= z - z_1 - \epsilon\,(z - z_1)\,(z - z_2) + \mathcal{O}(\epsilon^2).
\end{aligned} \tag{120}$$

The transformation (120), after composing with a uniform translation $z \to z + z_1$ to remove the second term, agrees with (119) to leading order. Thus we expect that the correlation function $\langle \mathcal{O}_p\mathcal{O}_q\rangle_{S^2}$ should be invariant under such a transformation, to linear order in ϵ.

[4] Of course, this transformation will not be in $PSL(2, \mathbb{C})$ unless $\gamma = 1$, but the sphere is actually invariant under all Mobius transformations.

We recall that, under a general infinitesimal conformal transformation $\delta z = \epsilon(z)$, $\delta \bar{z} = \bar{\epsilon}(\bar{z})$, a primary operator $\mathcal{O}_p$ with weights $(h_p, \tilde{h}_p)$ will transform as

$$\delta\mathcal{O}_p(z,\bar{z}) = -h_p\epsilon'(z)\mathcal{O}_p(z,\bar{z}) - \epsilon(z)\partial_z\mathcal{O}_p(z,\bar{z}) - \tilde{h}_p\bar{\epsilon}'(\bar{z})\mathcal{O}_p - \bar{\epsilon}(\bar{z})\partial_{\bar{z}}\mathcal{O}_p(z,\bar{z}). \tag{121}$$

In this case, we have

$$\begin{aligned}\epsilon(z) &= \epsilon\left(z^2 - (z_1+z_2)z + z_1 z_2\right),\\ \bar{\epsilon}(\bar{z}) &= \bar{\epsilon}\left(\bar{z}^2 - (\bar{z}_1+\bar{z}_2)\bar{z} + \bar{z}_1\bar{z}_2\right).\end{aligned} \tag{122}$$

We must be somewhat careful in using the functional forms (122) to compute the transformation (121). In particular, $\mathcal{O}_p(z_1,\bar{z}_1)$ is a function of z_1, not of z. The transformation of $\mathcal{O}_p$, then, is really

$$\begin{aligned}\delta\mathcal{O}_p = \Big[&- h_p\epsilon\left(2z-(z_1+z_2)\right)\mathcal{O}_p - \epsilon\left(z^2-(z_1+z_2)z+z_1z_2\right)\partial\mathcal{O}_p \\ &- \tilde{h}_p\bar{\epsilon}\left(2\bar{z}-(\bar{z}_1+\bar{z}_2)\right)\mathcal{O}_p - \bar{\epsilon}\left(\bar{z}^2-(\bar{z}_1+\bar{z}_2)\bar{z}+\bar{z}_1\bar{z}_2\right)\bar{\partial}\mathcal{O}_p\Big]_{z\to z_1,\bar{z}\to\bar{z}_1}.\end{aligned} \tag{123}$$

But at $z = z_1$ and $\bar{z} = \bar{z}_1$, the terms in (123) proportional to $\partial\mathcal{O}_p$ and $\bar{\partial}\mathcal{O}_p$ vanish by construction, while the terms proportional to $\mathcal{O}_p$ simply become

$$\delta\mathcal{O}_p = -h_p\epsilon\left(z_1-z_2\right)\mathcal{O}_p - \tilde{h}_p\bar{\epsilon}\left(\bar{z}_1-\bar{z}_2\right)\mathcal{O}_p \tag{124}$$

Likewise, applying the same reasoning to the change in $\mathcal{O}_q$ and using that it is a function of z_2, one finds

$$\delta\mathcal{O}_q = -h_q\epsilon\left(z_2-z_1\right)\mathcal{O}_q - \tilde{h}_q\bar{\epsilon}\left(\bar{z}_2-\bar{z}_1\right)\mathcal{O}_q. \tag{125}$$

As we have argued above, the change in the correlator,

$$\begin{aligned}\langle(\delta\mathcal{O}_p)\,\mathcal{O}_q + \mathcal{O}_p\,(\delta\mathcal{O}_q)\rangle &= \Big\langle \Big(-h_p\epsilon\left(z_1-z_2\right)\mathcal{O}_p - \tilde{h}_p\bar{\epsilon}\left(\bar{z}_1-\bar{z}_2\right)\mathcal{O}_p\Big)\,\mathcal{O}_q \\ &\qquad + \mathcal{O}_p\Big(-h_q\epsilon\left(z_2-z_1\right)\mathcal{O}_q - \tilde{h}_q\bar{\epsilon}\left(\bar{z}_2-\bar{z}_1\right)\mathcal{O}_q\Big)\Big\rangle \\ &= \Big\langle\Big(\left(-h_p+h_q\right)\left(z_1-z_2\right) + \Big(-\tilde{h}_p+\tilde{h}_q\Big)\left(\bar{z}_1-\bar{z}_2\right)\Big)\mathcal{O}_p\mathcal{O}_q\Big\rangle\end{aligned} \tag{126}$$

must vanish to leading order in ϵ. This can occur in only two ways: either $h_p = h_q$ and $\tilde{h}_p = \tilde{h}_q$, so that the quantities in parentheses vanish, or $\langle\mathcal{O}_p\mathcal{O}_q\rangle = 0$. But this dichotomy is precisely the content of equation (6.7.5), namely

$$\langle\mathcal{O}_p(z_1,\bar{z}_1)\mathcal{O}_q(z_2,\bar{z}_2)\rangle_{S^2} = 0 \quad \text{unless } h_p = h_q, \quad \tilde{h}_p = \tilde{h}_q. \tag{127}$$

Now we would like to confirm (6.7.6),

$$\left\langle\prod_{i=1}^{3}\mathcal{O}_{p_i}(z_i,\bar{z}_i)\right\rangle_{S^2} = C_{p_1p_2p_3}\prod_{\substack{i,j=1\\ i<j}}^{3} z_{ij}^{h-2(h_i+h_j)}\bar{z}_{ij}^{\tilde{h}-2(\tilde{h}_i+\tilde{h}_j)}. \tag{128}$$

The fastest way to verify this is to argue that it could not have been any other way. By translation invariance, the correlator (128) can only be a function of the differences z_{ij}, and it must be consistent with transformation law under re-scaling $z \to \gamma z$,

$$\mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3} \longmapsto \gamma^{-h_1-\tilde{h}_1}\gamma^{-h_2-\tilde{h}_2}\gamma^{-h_3-\tilde{h}_3}\mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}. \tag{129}$$

We see that the only possible function of the differences z_{ij} which obeys (129) is

$$\langle\mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\rangle \sim z_{12}^{-h_1-h_2+h_3}\bar{z}_{12}^{-\tilde{h}_1-\tilde{h}_2+\tilde{h}_3}z_{23}^{-h_2-h_3+h_1}\bar{z}_{23}^{-\tilde{h}_2-\tilde{h}_3+\tilde{h}_1}z_{13}^{-h_1-h_3+h_2}\bar{z}_{13}^{-\tilde{h}_1-\tilde{h}_3+\tilde{h}_2}, \tag{130}$$

which is the same as Polchinski's equation (6.7.6).

Another way to see this is to use our results from part (a). We will map the three points z_1, z_2, z_3 to $0, 1, \infty$ using the map

$$z \to z' = \frac{(z_2 - z_3)(z - z_1)}{(z_2 - z_1)(z - z_3)}. \tag{131}$$

By Mobius invariance, the correlator must be invariant under this map. The quantity $\langle \mathcal{O}_{p_1}(0)\mathcal{O}_{p_2}(1)\mathcal{O}_{p_3}(\infty)\rangle$ is some (infinite) constant independent of position, and we pick up a few extra factors using the transformation law for tensor operators under conformal maps. As above, let's be careful about the points where the derivatives are evaluated: the transformation law for the holomorphic part, $z \to z'(z)$, reads

$$\langle \mathcal{O}_{p_1}(z_1)\mathcal{O}_{p_2}(z_2)\mathcal{O}_{p_3}(z_3)\rangle = \left(\frac{\partial z'}{\partial z}\right)^{-h_1}_{z=z_1}\left(\frac{\partial z'}{\partial z}\right)^{-h_2}_{z=z_2}\left(\frac{\partial z'}{\partial z}\right)^{-h_3}_{z=z_3}\langle \mathcal{O}_{p_1}(z_1')\mathcal{O}_{p_2}(z_2')\mathcal{O}_{p_3}(z_3')\rangle, \tag{132}$$

with a similar rule for $\bar{z} \to \bar{z}'(\bar{z})$. The derivative of the map (131) with respect to z is

$$\frac{dz'}{dz} = \frac{(z_1 - z_3)(z_3 - z_2)}{(z_1 - z_2)(z - z_3)^2}. \tag{133}$$

Plugging in $z = z_3$ will give a divergence (or, after raising the result to the power $-h_3$, zero), but this can be re-absorbed into the overall infinite constant – it is simply an artifact of the choice to send z_3 to ∞ rather than some finite point, which makes the formulas easier to work with. But using the formula (133) in (132), absorbing the divergence into some constant

$$C_{p_1p_2p_3} = \lim_{z \to z_3, z_3 \to \infty}\left[\langle O_{p_1}(0,0)\mathcal{O}_{p_2}(1,1)\mathcal{O}_{p_3}(z_3,\bar{z}_3)\rangle\,(z - z_3)^{h_3}(\bar{z} - \bar{z}_3)^{\tilde{h}_3}\right], \tag{134}$$

Doing this, the remaining finite factors involving z_1, z_2, z_3 again agree with Polchinski's equation (6.7.6).

Finally, we consider (6.7.7),

$$\left\langle \prod_{i=1}^{4}\mathcal{O}_{p_i}(z_i,\bar{z}_i)\right\rangle_{S^2} = C_{p_1p_2p_3p_4}(z_c,\bar{z}_c)\,(z_{12}z_{34})^{h}\,(\bar{z}_{12}\bar{z}_{34})^{\tilde{h}} \times \prod_{\substack{i,j=1\\ i<j}}^{4} z_{ij}^{-h_i-h_j}\,\bar{z}_{ij}^{-\tilde{h}_i-\tilde{h}_j}. \tag{135}$$

Here we have defined $h = \sum_i h_i$, $\tilde{h} = \sum_i \tilde{h}_i$, and $z_c = \frac{z_{12}z_{34}}{z_{13}z_{24}}$.

We could handle this by using a Mobius transformation to fix three of the z_i, leaving the result in terms of one variable z_c, but a faster way is simply to argue by consistency with Mobius invariance, as we did in the first verification of (6.7.6) above.

Given four points z_i, it is possible to construct Mobius invariants called cross-ratios.[5]. There are six such cross-ratios, but they can all be expressed in terms of one another, so we may as well just pick one of them, say

$$z_c = \frac{z_{12}z_{34}}{z_{13}z_{24}}. \tag{136}$$

One can check that z_c is invariant under Mobius transformations[6].

Thus our first argument for (6.7.6) – that it is the only functional form consistent with the scaling properties, up to an overall constant – no longer holds here, since an arbitrary function of a Mobius invariant can still multiply the scaling factors. But this is the only ambiguity allowed, and it is precisely that captured by (135).

We conclude that Polchinski's (6.7.7) is indeed the most general allowed functional form for a four-point function of tensor operators, as desired.

[5] See page 170 of Peter West's "Introduction to Strings and Branes" for a lucid discussion about this.
[6] It is a function of differences, and is homogeneous of degree zero, and has the same index appearing as many times in the numerator as in the denominator; these three properties are enough to guarantee Mobius invariance.

Exercise 7. *Polchinski 7.2* 10 points
Derive the torus vacuum amplitude (7.3.6) by regulating and evaluating the determinants, as is done for the harmonic oscillator in appendix A. Show that a modular transformation just permutes the eigenvalues. [Compare exercise A.3.]

Solution 7.
Let's start from scratch. We are interested in computing

$$Z_{T^2} = \int_{F_0} \frac{d\tau\, d\bar{\tau}}{4\tau_2} \langle b(0)\tilde{b}(0)\tilde{c}(0)c(0)\rangle_{T^2} \langle 1\rangle_{X^\mu}, \tag{137}$$

which is Polchinski's equation (7.3.6), although he leaves the vacuum amplitude $\langle 1\rangle_{X^\mu}$ implicit.

First we'll think about the bosonic fields X^μ, and then we'll handle the bc ghosts later. The worldsheet action for the free scalars is

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-\det\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu. \tag{138}$$

We are interested in computing the partition function $\mathcal{Z} = \int \mathcal{D}X^\mu \mathcal{D}\gamma_{ab} e^{-S}$, which requires an integration over all possible field configurations X^μ and all metrics γ_{ab} on the torus (but *not* the sum over all worldsheet topologies – we are restricting to one-holed donuts).

As Polchinski told us in section 5.1, every metric on the torus can be brought to the form

$$ds^2 = \left|d\sigma^1 + \tau d\sigma^2\right|^2 \tag{139}$$

for a complex number τ called the *complex structure*. Integrating over all torus metrics, then, is equivalent to integrating over all inequivalent choices of τ, which means that we should integrate over the so-called *fundamental domain*.

More on that in a moment. For now, let's rewrite the action (138) for a given metric (i.e. a given τ), yielding

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left|\tau X'^\mu - \dot{X}^\mu\right|^2. \tag{140}$$

Here I write $\dot{X}^\mu = \frac{\partial X^\mu}{\partial\tau} = \frac{\partial X^\mu}{\partial\sigma_2}$, but please do not confuse the worldsheet time coordinate τ with the torus complex structure τ!

After an integration by parts (not there is no boundary, since both σ and τ are periodic on a toroidal worldsheet), the action (140) becomes

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma\, X^\mu \underbrace{\left(\frac{1}{\tau_2}\left|\tau\frac{\partial}{\partial\sigma_1} - \frac{\partial}{\partial\sigma_2}\right|^2\right)}_{\equiv\Box_{\text{torus}}} X_\mu. \tag{141}$$

Here we have defined the Laplacian on the torus,

$$\Box_{\text{torus}} = \frac{1}{\tau_2}\left|\tau\frac{\partial}{\partial\sigma_1}\frac{\partial}{\partial\sigma_2}\right|^2. \tag{142}$$

Again, $\tau = \tau_1 + i\tau_2$ is the complex structure and unrelated to the worldsheet coordinate σ_2.

Now we will do the path integral over all X^μ. The usual strategy for such things is to first find classical solutions of the equations of motion, and then integrate over all fluctuations around those classical solutions. In this case, the classical equation of motion on the torus is $\Box_{\text{torus}}X^\mu = 0$.

Aside: Interestingly, if we had a *compact* scalar X^μ, so that we identify the values $X^\mu \simeq X^\mu + 2\pi$, then we would find a space of doubly-periodic classical solutions on the torus which are completely determined by the two *winding numbers* n_1 and n_2 that count how many times the solution wraps around the two cycles of the torus:

$$X^\mu_{\text{classical}} \sim 2\pi n_1\sigma_1 + 2\pi n_2\sigma_2, \tag{143}$$

where $2\pi R_i$ are the circumferences of the two torus cycles.

However, for the case at hand, we have no such identification. Since X^μ needs to be periodic on the torus, the only allowed classical solution is a constant.

Next we need to integrate over fluctuations. Any given fluctuation δX^μ can be expanded in eigenfunctions of the Laplacian on the torus. These eigenfunctions are labeled by two integers n_1 and n_2 that describe the winding numbers of an exponential phase factor around the two cycles of the torus.

$$\begin{aligned} \delta X^\mu &= \sum_{n_1,n_2} c_{n_1,n_2} f_{n_1,n_2}, \\ \Box_{\text{torus}} f_{n_1,n_2} &= -\lambda_{n_1,n_2} f_{n_1,n_2}. \end{aligned} \tag{144}$$

Actually, we can find the eigenfunctions λ_n and their eigenvalues explicitly.

$$\begin{aligned} f_{n_1,n_2} &= \exp\left(2\pi i\left(n_1\sigma_1 + n_2\sigma_2\right)\right), \\ \lambda_{n_1,n_2} &= \frac{4\pi^2}{\tau_2}\left|n_1\tau - n_2\right|. \end{aligned} \tag{145}$$

Progress! Thus the action for the fluctuations, $S = \frac{1}{4\pi\alpha'}\int d^2\sigma\, X^\mu \Box_{\text{torus}} X_\mu$, becomes simply

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma \left(\sum_{n_1,n_2} c_{n_1,n_2} f_{n_1,n_2}\right)\left(-\lambda_{n_1,n_2}\right)\left(\sum_{m_1,m_2} c_{m_1,m_2} f_{m_1,m_2}\right). \tag{146}$$

We can collapse the integral over sums in (146) pretty easily, since eigenfunctions of the Laplacian are orthonormal as we might expect[7] :

$$\int d^2\sigma\, f_{n_1,n_2} f_{m_1,m_2} = \delta_{n_1,-m_1}\delta_{n_2,-m_2}. \tag{147}$$

So the action is just

$$S = \frac{1}{4\pi\alpha'}\sum_{n_1,n_2} \lambda_{n_1,n_2} A_{n_1,n_2} A_{-n_1,-n_2}. \tag{148}$$

If we assume that the fluctuations $\delta X^\mu = \sum_{n_1,n_2} c_{n_1,n_2} f_{n_1,n_2}$ are real quantities (as they should be, if they are labeling positions in the target spacetime!), then we obtain the reality condition $c_{-n_1,-n_2} = c^\star_{n_1,n_2}$, since the phases in the complex exponentials f_{n_1,n_2} of course flip sign when we conjugate. Using this, the product $A_{n_1,n_2}A_{-n_1,-n_2}$ is just the modulus $|A_{n_1,n_2}|^2$. Overall, then, our path integral over fluctuations looks like

$$\int \mathcal{D}\delta X^\mu \exp\left(-\frac{1}{4\pi\alpha'}\sum_{n_1,n_2}\lambda_{n_1,n_2}|A_{n_1,n_2}|^2\right), \tag{149}$$

where the λ's were defined in (145). This is starting to look like a product of Gaussian integrals[8], once we trade the path integral over $\mathcal{D}X^\mu$ for a product of ordinary integrals over the modes A_{n_1,n_2}. Doing this, we find

$$Z \sim \frac{1}{\prod_{n_1,n_2}\sqrt{\lambda_{n_1,n_2}}}. \tag{150}$$

The meat of this problem is regularizing the infinite product of the λ_{n_1,n_2}. As in problem 5.1, we can do this either via Pauli-Villars or zeta function regularization.

I prefer the zeta function approach. I will not include the details here; for the full calculation, see section 10.2 of Francesco's CFT book. The punchline is that

$$\prod_{n_1,n_2}\frac{4\pi^2}{\tau_2}\left|n_1\tau + n_2\right| = 4\pi^2\tau_2\eta^2(\tau)\bar{\eta}^2(\tau). \tag{151}$$

[7] Although note the minus signs in the delta functions – the eigenfunctions are orthonormal unless their indices *sum* to zero.

[8] This is quite fortuitous, since those are the only integrals I know how to do.

where we have used the form of the eigenvalues in (145).

We have argued above that the classical solutions X_0^μ on the torus are simply constant zero modes; integration over the zero mode will give another factor of $\sqrt{\tau}_2$.

Next we should evaluate the $\langle b(0)\tilde{b}(0)c(0)\tilde{c}(0)$ piece. We can do this in the same way, by reducing the integral to a functional determinant and then regularizing (this is described in section 6.3 of Dieter Lust's *Basic Concepts of String Theory*). The product of the fermionic determinants, once regularized, contributes another factor of $(\operatorname{Im}\tau)^2\,|\eta(\tau)|^4$.

Including all contributions from the matter and ghost fields, including the zero modes, we are left with -48 powers of $|\eta(\tau)|$ and -14 powers of $\operatorname{Im}(\tau)$ (including the one in the partition function integral which guarantees modular invariance of the measure), so that

$$Z_{T^2} \sim \int \frac{d\tau\, d\bar{\tau}}{\tau_2} \tau_2^{-13} |\eta(\tau)|^{-48}, \tag{152}$$

as claimed in Polchinski's equation (7.3.6).

Exercise 8. *Polchinski 7.4* 10 points

Evaluate $\langle \partial_w X^\mu(w) \partial_w X_\mu(0) \rangle$ on the torus by representing it as a trace. Show that the result agrees with equation (7.2.16).

Solution 8.

I will write the desired correlator as

$$[\langle \partial_w X^\mu(w) \partial_z X_\mu(z) \rangle_{T^2}]_{z=0} = [\partial_w \partial_z \langle X^\mu(w) X_\mu(z) \rangle_{T^2}]_{z=0} \tag{153}$$

to emphasize that we are computing it on the torus. Our strategy will be to first compute the quantity $\langle X^\mu(w) X_\mu(z) \rangle_{T^2}$ on the right side of (153), which is simply the propagator on the torus, and then take derivatives.

As we have seen in section 7.2 of Polchinski, correlators on a torus with modular parameter τ can be represented in terms of correlators on a spatial circle by time-evolving for $2\pi\tau_2$ (using the operator H, which generates time translations) and translating in space by $2\pi\tau_1$ (using the momentum operator P to generate spatial translations), and then identifying the ends (gluing the circles together is implemented by a trace over Hilbert space states). Operationally, this means that

$$\begin{aligned} \langle X^\mu(\sigma) X_\mu(\sigma') \rangle_{T^2} &= \mathrm{Tr}\left[X^\mu(\sigma) X_\mu(\sigma') e^{2\pi i \tau_1 P - 2\pi\tau_2 H} \right] \\ &= \mathrm{Tr}\left[X^\mu(\sigma) X_\mu(\sigma') q^{L_0} \bar{q}^{\bar{L}_0} \left(q\bar{q}\right)^{-d/24} \right]. \end{aligned} \tag{154}$$

Here I have gone back to real worldsheet coordinates σ_0, σ_1 rather than complex coordinates z, $\bar{z}$ to avoid confusing myself – our calculation is still on the cylinder rather than the plane because we are gluing the cylinder end-caps together.

Now, we can expand the two insertions of $X^\mu(\sigma)$ in terms of oscillators as

$$X^\mu(\sigma_0, \sigma_1) = x^\mu + \alpha' p^\mu \sigma_0 + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left(\frac{\alpha_n^\mu}{n} e^{-in(\sigma_0 - \sigma_1)} + \frac{\tilde{\alpha}_n^\mu}{n} e^{-in(\sigma_0 + \sigma_1)} \right). \tag{155}$$

Doing this, we find

$$\begin{aligned} \langle X^\mu(\sigma) X_\mu(\sigma') \rangle_{T^2} = \mathrm{Tr}\Bigg[&\left(x^\mu + \alpha' p^\mu \sigma_0 + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left(\frac{\alpha_n^\mu}{n} e^{-in(\sigma_0 - \sigma_1)} + \frac{\tilde{\alpha}_n^\mu}{n} e^{-in(\sigma_0 + \sigma_1)} \right) \right) \\ &\cdot \left(x_\mu + \alpha' p_\mu \sigma_0' + i\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \left(\frac{\alpha_{m,\mu}}{m} e^{-im(\sigma_0' - \sigma_1')} + \frac{\tilde{\alpha}_{m,\mu}}{m} e^{-im(\sigma_0' + \sigma_1')} \right) \right) q^{L_0} \bar{q}^{\bar{L}_0} \left(q\bar{q}\right)^{-d/24} \Bigg]. \end{aligned} \tag{156}$$

The trace (156) should run over all possible string states. I will use the notation $|\Lambda\rangle$, where $\Lambda = \{\Lambda_{\mu,n}\}$ is a multi-index, to refer to a string state that has $\Lambda_{\mu,n} \in \mathbb{N}$ oscillator excitations in the n-th level of the μ-th direction:

$$|\Lambda\rangle = \prod_{\mu=0}^{d} \prod_{n=1}^{\infty} \frac{\left(\alpha_{-n}^\mu\right)^{\Lambda_{\mu,n}}}{\sqrt{\left(n^{\Lambda_{\mu,n}}\right)\left(\Lambda_{\mu,n}!\right)}} |0\rangle. \tag{157}$$

Said differently, if a state $|\Lambda\rangle$ has λ_n excitations in the n level, we have

$$\begin{aligned} \alpha_n^\dagger |\Lambda\rangle = \alpha_{-n}|\Lambda\rangle &= \sqrt{n}\sqrt{\lambda_n + 1} |\{\lambda_1, \cdots, \lambda_{n-1}, \lambda_n + 1, \lambda_{n+1}, \cdots, \}, \\ \alpha_n |\Lambda\rangle &= \sqrt{n}\sqrt{\lambda_n} |\{\lambda_1, \cdots, \lambda_{n-1}, \lambda_n - 1, \lambda_{n+1}, \cdots, \}, \end{aligned} \tag{158}$$

which looks like the usual rules for the harmonic oscillator, except with a scale factor of $\sqrt{n}$.

The term in (156) proportional to $x_\mu x^\mu$ is an overall constant which will disappear when we later differentiate, so let's ignore it for now. The momentum term can be reduced to oscillators using

$p^\mu p_\mu = -M^2 = \frac{4}{\alpha'} \sum_{i>0} \alpha_i \alpha_{-i}$, which we will return to shortly. For now let's focus on the oscillator terms. We want to understand how to evaluate traces like

$$\sum_{\Lambda,m,n} \left\langle \Lambda \middle| \frac{\alpha_n^\mu}{n} e^{-in(\sigma_0-\sigma_1)} \frac{\alpha_{m,\mu}}{m} e^{-im(\sigma_0'-\sigma_1')} q^{L_0} \bar{q}^{\bar{L}_0} (q\bar{q})^{-d/24} \middle| \right\rangle, \tag{159}$$

since the calculations for the tilde'd oscillators will be identical.

Only some of the terms in (159) will survive. Any string state $|\Lambda\rangle$ is an eigenstate of q^{L_0}, but the oscillators $\alpha_{\mu,n}$ and α_m^μ will generically act on a state $|\Lambda\rangle$ to give some orthogonal state by adding or removing oscillator excitations. The only way the resulting state will have nonzero inner product with the $\langle\Lambda|$ on the left is if $n = -m$. Hence the remaining terms are proportional to

$$\sim \sum_{\Lambda,n} \left\langle \Lambda \middle| \frac{\alpha_n^\mu}{n} e^{-in(\sigma_0-\sigma_1)} \frac{\alpha_{-n,\mu}}{n} e^{in(\sigma_0'-\sigma_1')} q^{L_0} \middle| \Lambda \right\rangle, \tag{160}$$

In (160), I have dropped the $(q\bar{q})^{-d/24}$ factor and the insertion of $\bar{q}^{\bar{L}_0}$ for simplicity; let's focus on only the un-tilde'd oscillators for the moment.

Review of q Trace Computation.

To evaluate (160), it's worth reminding ourselves of how we evaluates $\sum_\Lambda \left\langle \Lambda \middle| q^{L_0} \middle| \Lambda \right\rangle$, with no oscillator insertions, in the partition function calculation.[9] In that context, we noted that $q^{L_0} = q^{\sum_n \alpha_{-n}\alpha_n}$ and simplified the problem by considering each family of oscillators separately. For instance, in a basis of states that only excite the n oscillator – i.e., a basis $|\lambda_n = 0\rangle$, $|\lambda_n = 1\rangle$, $|\lambda_n = 2\rangle$, etc. – the operator $q^{\alpha_{-n}\alpha_n}$ acts as

$$q^{\alpha_{-n}\alpha_n} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & q^n & 0 & 0 & \cdots \\ 0 & 0 & q^{2n} & 0 & \cdots \\ 0 & 0 & 0 & q^{3n} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{161}$$

Thus, in this subspace, we see that $\mathrm{Tr}(q^{\alpha_{-n}\alpha_n}) = \sum_{i=0}^\infty q^{in} = \frac{1}{1-q^n}$.

But we can decouple the spaces of oscillator excitations, since

$$\mathrm{Tr}\left(q^{\sum_{n=0}^\infty \alpha_{-n}\alpha_n}\right) = \prod_{n=0}^\infty \mathrm{Tr}\left(q^{\alpha_{-n}\alpha_n}\right) = \prod_{n=0}^\infty \frac{1}{1-q^n}, \tag{162}$$

which is how we obtained the dependence on the Dedekind eta function $\eta(q) = q^{\frac{1}{24}} \prod_{n=0}^\infty (1-q^n)^{-1}$ in the ordinary partition function calculation.

Back to the problem at hand.

We will have to be somewhat more careful in tracking different terms due to the oscillator insertions in our problem. For instance, fix one index $n > 0$ in the sum, and consider the trace over all multi-indices Λ. This is

$$\begin{aligned} &\sum_{\{m_i\}} \left\langle \lambda_1 = m_1, \lambda_2 = m_2, \cdots \middle| \frac{\alpha_n}{n} e^{-in(\sigma_0-\sigma_1)} \frac{\alpha_{-n}}{n} e^{in(\sigma_0'-\sigma_1')} q^{L_0} \middle| \lambda_1 = m_1, \lambda_2 = m_2, \cdots \right\rangle \\ &= e^{-in\left((\sigma_0'-\sigma_0)-(\sigma_1'-\sigma_1)\right)} \frac{1}{n^2} \sum_{\{m_i\}} \left(q^{1m_1} q^{2m_2} q^{3m_3\cdots}\right) \underbrace{\left\langle \lambda_1 = m_1, \lambda_2 = m_2, \cdots \middle| \alpha_n\alpha_{-n} \middle| \lambda_1 = m_1, \lambda_2 = m_2, \cdots \right\rangle}_{n(m_n+1)} \\ &= e^{-in\left((\sigma_0'-\sigma_0)-(\sigma_1'-\sigma_1)\right)} \frac{1}{n} \sum_{\{m_i\}} \left(q^{nm_n}(m_n+1)\right) \left(\prod_{m_j \neq m_n} q^{jm_j}\right) \end{aligned} \tag{163}$$

[9]See Insert 3.4 in Clifford Johnson's *D-Branes* for a better explanation of this.

For all of the indices $m_j \neq m_n$ in the last line of (163), we can split the sum over multi-indices into sums over each m_j and repeat the same calculation as in the case with no insertions. In each case, we have the geometric sum $q^0 + q^j + q^{2j} + \cdots = \frac{1}{1-q^j}$. So

$$\sum_{\{m_i\}} \left(q^{nm_n} (m_n + 1) \right) \left(\prod_{m_j \neq m_n} q^{jm_j} \right) = \left(\sum_{m_n = 0}^{\infty} q^{nm_n} (m_n + 1) \right) \left(\prod_{j \neq n} \frac{1}{1 - q^j} \right) \tag{164}$$

The first term can be split into two sums. We evaluate them as

$$\sum_{m_n = 1}^{\infty} q^{nm_n} m_n = \frac{q^n}{(1 - q^n)^2}, \tag{165}$$

which can be proven by differentiating the usual geometric sum formula, and the second is simply the geometric sum $\sum_{m_n=0}^{\infty} q^{mn} = \frac{1}{1-q^n}$.

All in all, then, we have found that the trace for a term with fixed $n > 0$ gives a contribution

$$e^{-in((\sigma_0' - \sigma_0) - (\sigma_1' - \sigma_1))} \frac{1}{n} \left(\prod_{j \neq n} \frac{1}{1 - q^j} \right) \left(\frac{q^n}{(1 - q^n)^2} + \frac{1}{1 - q^n} \right). \tag{166}$$

Let's factor out one copy of $\frac{1}{1-q^n}$ from the final factor and insert it into the product, which now runs over all j rather than simply $j \neq n$. Next, recall that this trace was only for a single term $\alpha_n \alpha_{-n}$ with $n > 0$. We still need to sum over all n to find the final contribution to the propagator. So we really want to evaluate

$$\sum_{n=1}^{\infty} e^{-in((\sigma_0' - \sigma_0) - (\sigma_1' - \sigma_1))} \frac{1}{n} \left(\prod_{j} \frac{1}{1 - q^j} \right) \left(\frac{q^n}{1 - q^n} + 1 \right). \tag{167}$$

It will be convenient to re-express the result using the formulas

$$\sum_{n=1}^{\infty} \frac{e^{inx}}{n} = - \log \left(1 - e^{ix} \right), \tag{168}$$

and

$$\sum_{m=1}^{\infty} \frac{1}{m} \frac{x^m}{1 - y^m} = - \sum_{n=0}^{\infty} \log \left(1 - x y^n \right), \tag{169}$$

the latter of which is equation (8.1.32) in volume 2 of Green, Schwartz, and Witten's string theory book. For convenience, let's also define $\Delta\sigma = (\sigma_0' - \sigma_0) - (\sigma_1' - \sigma_1)$. Then the sum (167) becomes

$$\sum_{n=1}^{\infty} e^{-in\Delta\sigma} \frac{1}{n} \left(\prod_j \frac{1}{1 - q^j} \right) \left(\frac{q^n}{1 - q^n} + 1 \right) = \left(\prod_j \frac{1}{1 - q^j} \right) \cdot \left[- \log \left(1 - e^{-i\Delta\sigma} \right) - \sum_{m=0}^{\infty} \log \left(1 - q^{m+1} e^{-i\Delta\sigma} \right) \right]. \tag{170}$$

Good, we've found half of the un-tilde'd oscillator sum! Next we also need to sum over the negative values of n. As before, fix a particular $n < 0$ and consider the trace over all multi-particle indices Λ. This time, the operator α_{-n} which hits the state after q^{L_0} is a *lowering* operator, so the product $\alpha_n \alpha_{-n}$ brings out a factor of $(-n)\, m_{-n}$ rather than $n(m_n + 1)$.

$$\begin{aligned}
&\sum_{\{m_i\}} \left\langle \lambda_1 = m_1, \lambda_2 = m_2, \cdots \middle| \frac{\alpha_n}{n} e^{-in(\sigma_0 - \sigma_1)} \frac{\alpha_{-n}}{n} e^{in(\sigma_0' - \sigma_1')} q^{L_0} \middle| \lambda_1 = m_1, \lambda_2 = m_2, \cdots \right\rangle \\
&= e^{-in\Delta\sigma} \frac{1}{n^2} \sum_{\{m_i\}} \left(q^{1m_1} q^{2m_2} q^{3m_3 \cdots} \right) \underbrace{\left\langle \lambda_1 = m_1, \lambda_2 = m_2, \cdots \middle| \alpha_n \alpha_{-n} \middle| \lambda_1 = m_1, \lambda_2 = m_2, \cdots \right\rangle}_{-nm_{-n}} \\
&= - e^{-in\Delta\sigma} \frac{1}{n} \sum_{\{m_i\}} \left(q^{-nm_{-n}} m_{-n} \right) \left(\prod_{m_j \neq m_{-n}} q^{jm_j} \right) = - e^{-in\Delta\sigma} \frac{1}{n} \left(\frac{q^{-n}}{(1 - q^{-n})^2} \right) \left(\prod_{j \neq -n} \frac{1}{1 - q^j} \right) \\
&= - e^{-in\Delta\sigma} \frac{1}{n} \left(\frac{q^{-n}}{1 - q^{-n}} \right) \left(\prod_j \frac{1}{1 - q^j} \right).
\end{aligned} \tag{171}$$

Same deal: we still need to sum over all negative values of n by re-labeling $n \to -n$ and then using (169) to sum over all positive n. Doing so, (171) becomes

$$\left(\prod_j \frac{1}{1-q^j}\right) \cdot \left[-\sum_{m=0}^{\infty} \log\left(1 - q^{m+1} e^{i\Delta\sigma}\right) \right]. \tag{172}$$

We're still not done. There will be a contribution from the momentum term, since $p^\mu p_\mu = \frac{4}{\alpha'} \sum_{i>0} \alpha_i \alpha_{-i}$. This calculation is very similar to that in equation (163), except that there are no factors of $e^{-in\Delta\sigma}$ or $\frac{1}{n^2}$. We can see the result from stripping the exponential factor off of equation (166) and replacing the factor of n^2 to find

$$\left(\prod_j \frac{1}{1-q^j}\right) \left(\frac{nq^n}{1-q^n} + n\right). \tag{173}$$

When we sum over n, the second term in parentheses in (173) is divergent and must be regularized. We might have expected this, since we are taking a trace with an insertion of an operator that is proportional to mass-squared, and naturally our Hilbert space has arbitrarily massive states. We can use zeta function regularization, since of course

$$\sum_{n=1}^{\infty} n = \zeta(-1) = -\frac{1}{12}. \tag{174}$$

For the first term, recall that $q = e^{2\pi i \tau}$, so we can write this term as

$$\begin{aligned} -\frac{1}{2\pi i} \partial_\tau \log\left(1 - q^n\right) &= -\frac{1}{2\pi i} \frac{1}{1-q^n} \partial_\tau \left(1 - e^{2\pi i \tau n}\right) \\ &= \frac{nq^n}{1-q^n}. \end{aligned} \tag{175}$$

Now we have showed all of the traces for the un-tilde'd oscillators and for the momentum contribution. There will also be the corresponding contributions from the tilde'd oscillators, which are identical to those above except that they have all instances of $\sigma_0 + \sigma_1$ replaced by $\sigma_0 - \sigma_1$ (when we switch to complex coordinates, this will simply mean replacing z by $\bar{z}$). We have also suppressed the Lorentz index μ, but this should of course run over all dimensions (I think it suffices to work in light-cone gauge, so that there are only 24 transverse directions, and ignore the contributions of ghosts, but I might be wrong).

You may worry about the factors of $\left(\prod_j \frac{1}{1-q^j}\right)$ that multiply all of our logarithms. These products, proportional to $\eta(\tau)$, are independent of the worldsheet coordinates σ_i, so they represent an arbitrary normalization. If we divide by the partition function, as Polchinski does in the equation (7.2.16) which we seek to verify,

$$Z(\tau)^{-1} \langle \partial_w X^\mu(w) \partial_w X_\mu(0) \rangle, \tag{176}$$

then these factors will cancel out.

I will skip some steps involving tracking constants carefully and give a flavor for the rest of the calculation. Our oscillator traces gave us expressions of the form

$$\sum_{m=1}^{\infty} \log\left(1 - q^m e^{i\Delta\sigma}\right) + \log\left(1 - q^m e^{-i\Delta\sigma}\right) \tag{177}$$

in the propagator. Since we are ultimately going to take derivatives with respect to w, we can add any w-independent function of τ without changing the result. Let's add a term $\log(1-q^m)$ to the sum, which gives

$$\begin{aligned} &\sum_{m=1}^{\infty} \left[\log\left(1 - q^m e^{i\Delta\sigma}\right) + \log\left(1 - q^m e^{-i\Delta\sigma}\right) + \log(1-q^m)\right] \\ &\quad = \log\left[\prod_{m=1}^{\infty} \left(1 - q^m e^{i\Delta\sigma}\right)\left(1 - q^m e^{-i\Delta\sigma}\right)(1-q^m)\right] \end{aligned} \tag{178}$$

Compare this to the infinite product representation of the first Jacobi theta function, namely

$$\theta_1(\nu|\tau) = 2\exp\left(\frac{\pi i\tau}{4}\right)\sin(\pi\nu)\prod_{m=1}^{\infty}(1-q^m)(1-zq^m)(1-z^{-1}q^m), \tag{179}$$

where $z = e^{2\pi i\nu}$, which is given in Polchinski's equation (7.2.38d). We see that the expression (178) is $\log(\theta_1(\nu|\tau)) - \log(\sin(\pi\nu)) + f(\tau)$, where $f(\tau)$ is independent of w and thus will vanish when we take derivatives, if we choose $z = e^{i\Delta\sigma}$ and thus $\nu = \frac{\Delta\sigma}{2\pi}$.

When we include the tilde'd oscillators, we will get another factor that looks like $\log\left(\bar{\theta}_1\right)$, so the two conspire to give a term $\log\left(|\theta_1|^2\right)$. After tracking all of the contributions carefully, one can convince oneself that the overall propagator can be expressed in terms of these theta functions as

$$Z^{-1}\langle X(z)X(w)\rangle = -\frac{\alpha'}{2}\log\left[\left|\theta_1\left(\frac{z-w}{2\pi}|\tau\right)\right|^2\right] + \alpha'\frac{\mathrm{Im}((z-w)^2)}{4\pi\tau_2} + f(\tau). \tag{180}$$

which agrees with Polchinski's equation (7.2.3).

To conclude, we can find the desired quantity by taking two derivatives of (180) with respect to z and w, then setting $w = 0$. The result is precisely Polchinski's equation (7.2.16).